Population Ethics and the Existence of Value Functions

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Abstract. This note provides a corrected version of a classical result in population ethics. While it is true that the fixed-population axioms continuity and weak Pareto are sufficient for the existence of an ordering of population-size – representative-utility pairs that can be used to rank social alternatives, we show that, in order to obtain the existence of a real-valued representation, continuity must be strengthened. *Journal of Economic Literature* Classification Number: D63.

Keywordsfi Population Ethics, Value Functions, Representations.

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1. Introduction

Investigations in population ethics are primarily concerned with establishing a social ordering of alternatives which may have different populations and population sizes. Welfarist population principles use information about the lifetime well-being of those alive in different alternatives to order them and employ social-evaluation orderings of utility vectors of variable dimension. Typically, these principles satisfy anonymity: individuals are treated impartially without regard to their identities. We implicitly employ a weak anonymity condition by defining social-evaluation orderings on a set of utility vectors without keeping track of the individuals who experience them.

For any utility vector, representative utility is that level of utility which, if assigned to each person, produces a vector of the same dimension that is as good as the original one. In Blackorby and Donaldson [1984], it is shown that, if some weak conditions are imposed on fixed-population comparisons, knowledge of representative utilities and population sizes is sufficient to rank any two utility vectors; that is, there exists an ordering of population-size – representative-utility pairs that can be used to generate the underlying social-evaluation ordering. The result in Blackorby and Donaldson [1984] also states that the ordering of those pairs has a real-valued representation. This is not the case for all orderings satisfying the fixed-population axioms. This note presents a corrected version of the Blackorby-Donaldson claim using a strengthened version of continuity and provides an example showing that a real-valued representation need not exist under the original set of assumptions.

2. Definitions

The set of positive integers is denoted by $\mathcal{Z}_{++}$. Let $\mathcal{R}$ be the set of all real numbers and, for $n \in \mathcal{Z}_{++}$, let $\mathcal{R}^n$ be the n-fold Cartesian product of $\mathcal{R}$. $\mathbf{1}_n$ is the vector consisting of n ones. For $n \in \mathcal{Z}_{++}$ and $u, v \in \mathcal{R}^n$, $u \gg v$ if and only if $u_i > v_i$ for all $i \in \{1, \ldots, n\}$. A function $\Xi^n : \mathcal{R}^n \to \mathcal{R}$ is weakly increasing if and only if, for all $u, v \in \mathcal{R}^n$, $u \gg v$ implies $\Xi^n(u) > \Xi^n(v)$.

A social-evaluation ordering is a reflexive, transitive, and complete binary relation R on $\Omega = \cup_{n \in \mathcal{Z}_{++}} \mathcal{R}^n$ where, for all $n, m \in \mathcal{Z}_{++}$, for all $u \in \mathcal{R}^n$, and for all $v \in \mathcal{R}^m$, uRv means that the utility vector u is socially at least as good as the utility vector v. The asymmetric and symmetric factors of R are denoted by P and I. The following fixed-population axioms are imposed on R.

Continuity: For all $n \in \mathcal{Z}_{++}$ and for all $u \in \mathcal{R}^n$, the sets $\{v \in \mathcal{R}^n \mid vRu\}$ and $\{v \in \mathcal{R}^n \mid uRv\}$ are closed.

Weak Pareto: For all $n \in \mathcal{Z}_{++}$ and for all $u, v \in \mathcal{R}^n$, if $u \gg v$, then uPv.

3. Results

The representative utility $\xi \in \mathcal{R}$ corresponding to any vector $u \in \mathcal{R}^n$ with $n \in \mathcal{Z}_{++}$ is implicitly defined by

$$\xi \mathbf{1}_n I u. \tag{1}$$

Continuity and weak Pareto imply that representative utility exists and is unique for all $u \in \Omega$ and, as a consequence, there exists a sequence of representative-utility functions $\Xi^n \colon \mathcal{R}^n \to \mathcal{R}$ such that, for all $n \in \mathcal{Z}_{++}$ and for all $u, v \in \mathcal{R}^n$,

$$uRv \Longleftrightarrow \Xi^n(u) \geq \Xi^n(v) \tag{2}$$

where Ξ^n is continuous and weakly increasing and satisfies $\Xi^n(\gamma \mathbf{1}_n) = \gamma$ for all $\gamma \in \mathcal{R}$ (see, for example, Blackorby, Bossert, and Donaldson [2000] for a proof). Knowledge of the representative utilities of two utility vectors of the same dimension is sufficient to rank them.

For variable-population comparisons, the only information that is required in addition to representative utility is population size. This is the essence of Blackorby and Donaldson's [1984] result which we state as Theorem 1 below. Note that no assumptions other than the fixed-population axioms continuity and weak Pareto are required.

Theorem 1: *If R satisfies continuity and weak Paretofithen there exists an ordering* $\mathbf{R}$ *on $\mathcal{Z}_{++} \times \mathcal{R}$ such thatfifor all $n \in \mathcal{Z}_{++}$ and for all $\xi, \zeta \in \mathcal{R}$fi*

$$(n, \xi)\mathbf{R}(n, \zeta) \Longleftrightarrow \xi \geq \zeta \tag{3}$$

andfifor all $n, m \in \mathcal{Z}_{++}$fifor all $u \in \mathcal{R}^n$fiand for all $v \in \mathcal{R}^m$fi

$$uRv \Longleftrightarrow (n, \Xi^n(u))\mathbf{R}(m, \Xi^m(v)). \tag{4}$$

Proof. The existence of the representative-utility functions Ξ^n for all $n \in \mathcal{Z}_{++}$ follows from continuity and weak Pareto (see Blackorby, Bossert, and Donaldson [2000, Theorem 3]). Now define the relation $\mathbf{R}$ on $\mathcal{Z}_{++} \times \mathcal{R}$ by letting, for all $(n, \xi), (m, \zeta) \in \mathcal{Z}_{++} \times \mathcal{R}$,

$$(n, \xi)\mathbf{R}(m, \zeta) \Longleftrightarrow \xi \mathbf{1}_n R \zeta \mathbf{1}_m. \tag{5}$$

Clearly, $\mathbf{R}$ is an ordering. Because $\Xi^n(\gamma \mathbf{1}_n) = \gamma$ for all $n \in \mathcal{Z}_{++}$ and for all $\gamma \in \mathcal{R}$, it follows that $uI\Xi^n(u)\mathbf{1}_n$ for all $n \in \mathcal{Z}_{++}$ and for all $u \in \mathcal{R}^n$. Together with (5) and the transitivity of R, (4) follows. (3) is a consequence of weak Pareto. ■

We use $\mathbf{P}$ and $\mathbf{I}$ to denote the asymmetric and symmetric factors of $\mathbf{R}$ (defined in Theorem 1). The ordering $\mathbf{R}$ is represented by the function $W: \mathcal{Z}_{++} \times \mathcal{R} \to \mathcal{R}$ if and only if, for all $(n, \xi), (m, \zeta) \in \mathcal{Z}_{++} \times \mathcal{R}$,

$$(n, \xi)\mathbf{R}(m, \zeta) \Longleftrightarrow W(n, \xi) \geq W(m, \zeta). \tag{6}$$

Some of these orderings do not have a representation. Consider, for example, the lexicographic ordering defined by

$$(n, \xi)\mathbf{R}(m, \zeta) \Longleftrightarrow \xi > \zeta \text{ or } [\xi = \zeta \text{ and } n \geq m] \tag{7}$$

for all $(n, \xi), (m, \zeta) \in \mathcal{Z}_{++} \times \mathcal{R}$. This ordering does not have a real-valued representation. The proof of this claim is analogous to the proof of the corresponding observation for lexicographic orderings on $\mathcal{R}^2$. It is important that the continuous variable ξ has lexicographic priority over the discrete variable n in this example. The lexicographic ordering obtained by reversing priorities, given by

$$(n, \xi)\mathbf{R}(m, \zeta) \Longleftrightarrow n > m \text{ or } [n = m \text{ and } \xi \geq \zeta] \tag{8}$$

for all $(n, \xi), (m, \zeta) \in \mathcal{Z}_{++} \times \mathcal{R}$, does have a representation. The function $W: \mathcal{Z}_{++} \times \mathcal{R} \to \mathcal{R}$ defined by

$$W(n, \xi) = \begin{cases} n - 1 + \frac{1}{2}e^{\xi} & \text{if } \xi \leq 0, \\ n - \frac{1}{2}e^{-\xi} & \text{if } \xi > 0 \end{cases} \tag{9}$$

for all $(n, \xi) \in \mathcal{Z}_{++} \times \mathcal{R}$ represents the ordering defined in (8).

The existence of a value function can be guaranteed by strengthening continuity to the following variable-population version.

Extended Continuity: For all $n, m \in \mathcal{Z}_{++}$ and for all $u \in \mathcal{R}^n$, the sets $\{v \in \mathcal{R}^m \mid vRu\}$ and $\{v \in \mathcal{R}^m \mid uRv\}$ are closed.

If extended continuity obtains, a real-valued representation, continuous and increasing in its second argument, exists. In order to prove this result, we make use of the sets $\mathcal{I}^n_m$ defined by

$$\mathcal{I}^n_m = \{\xi \in \mathcal{R} \mid \exists \zeta \in \mathcal{R} \text{ such that } (n,\xi)\mathbf{I}(m,\zeta)\} \tag{10}$$

for all $n, m \in \mathcal{Z}_{++}$ with $n \neq m$. $\mathcal{I}^n_m$ is the set of representative utilities for a population of size n which correspond to representative utilities for population size m such that the two population-size – representative-utility pairs are indifferent.

Lemma 1 provides several results on the above sets that are used in the proof of Theorem 2.

Lemma 1: *If R satisfies weak Pareto and extended continuityfithenfifor all $n, m \in \mathcal{Z}_{++}$ with $n \neq m$ such that $\mathcal{I}^n_m \neq \emptyset$ and $\mathcal{I}^m_n \neq \emptyset$fifüfi$\mathcal{I}^n_m$ is an open intervalfiüifi at most one of $\mathcal{I}^n_m$ and $\mathcal{I}^m_n$ is bounded abovefi and füiifi at most one of $\mathcal{I}^n_m$ and $\mathcal{I}^m_n$ is bounded belowfi*

Proof. The existence of the ordering $\mathbf{R}$ on $\mathcal{Z}_{++} \times \mathcal{R}$ satisfying (3) and (4) follows from Theorem 1. By extended continuity, $\mathbf{R}$ is continuous in the sense that, for all $n, m \in \mathcal{Z}_{++}$ and for all $\xi \in \mathcal{R}$, the sets $\{\zeta \in \mathcal{R} \mid (m,\zeta)\mathbf{R}(n,\xi)\}$ and $\{\zeta \in \mathcal{R} \mid (n,\xi)\mathbf{R}(m,\zeta)\}$ are closed. It follows that, if $\xi \notin \mathcal{I}^n_m$ then

$$[(n,\xi)\mathbf{P}(m,\zeta) \text{ for all } \zeta \in \mathcal{R}] \;\text{ or }\; [(m,\zeta)\mathbf{P}(n,\xi) \text{ for all } \zeta \in \mathcal{R}]. \tag{11}$$

Suppose that $\mathcal{I}^n_m$ is not connected. Then there exist $\hat{\xi}, \mathring{\xi}, \tilde{\xi} \in \mathcal{R}$ such that $\hat{\xi} > \mathring{\xi} > \tilde{\xi}$, $\hat{\xi}, \tilde{\xi} \in \mathcal{I}^n_m$, and $\mathring{\xi} \notin \mathcal{I}^n_m$. Therefore, there exist $\hat{\zeta}, \tilde{\zeta} \in \mathcal{R}$ such that $(n,\hat{\xi})\mathbf{I}(m,\hat{\zeta})$ and $(n,\tilde{\xi})\mathbf{I}(m,\tilde{\zeta})$. Consequently, $(m,\hat{\zeta})\mathbf{P}(n,\mathring{\xi})$ and (11) implies that $(m,\zeta)\mathbf{P}(n,\mathring{\xi})$ for all $\zeta \in \mathcal{R}$. At the same time, $(n,\mathring{\xi})\mathbf{P}(m,\tilde{\zeta})$ and (11) implies that $(n,\mathring{\xi})\mathbf{P}(n,\zeta)$ for all $\zeta \in \mathcal{R}$, a contradiction. Consequently, $\mathcal{I}^n_m$ is connected.

Now suppose that both $\mathcal{I}^n_m$ and $\mathcal{I}^m_n$ are bounded above. Because these sets are nonempty, there exist $\check{\xi} \notin \mathcal{I}^n_m$, $\check{\zeta} \notin \mathcal{I}^m_n$ and $\overset{*}{\xi}, \overset{*}{\zeta} \in \mathcal{R}$ such that $\check{\xi} > \overset{*}{\xi}$, $\check{\zeta} > \overset{*}{\zeta}$, and $(n,\overset{*}{\xi})\mathbf{I}(m,\overset{*}{\zeta})$. An argument similar to the one above establishes that $(n,\check{\xi})\mathbf{P}(m,\zeta)$ for all $\zeta \in \mathcal{R}$ and $(m,\check{\zeta})\mathbf{P}(n,\xi)$ for all $\xi \in \mathcal{R}$, a contradiction. Therefore, at most one of $\mathcal{I}^n_m$ and $\mathcal{I}^m_n$ is bounded above. A similar argument establishes (iii).

Now suppose that $\mathcal{I}^n_m$ is not open. In that case it must be bounded above with $\bar{\xi} = \sup\{\xi \mid \xi \in \mathcal{I}^n_m\} \in \mathcal{I}^n_m$ or below with $\underline{\xi} = \inf\{\xi \mid \xi \in \mathcal{I}^n_m\} \in \mathcal{I}^n_m$. Suppose that the first of these holds. Then, there exists $\bar{\zeta} \in \mathcal{R}$ such that $\bar{\zeta} \geq \zeta$ for all $\zeta \in \mathcal{I}^m_n$ and $(m,\bar{\zeta})\mathbf{I}(n,\bar{\xi})$. This implies that both $\mathcal{I}^n_m$ and $\mathcal{I}^m_n$ are bounded above, contradicting (ii). A similar argument rules out the other option and it follows that $\mathcal{I}^n_m$ is open. ■

Now we can prove

Theorem 2: *If R satisfies extended continuity and weak Paretofithen there exists a value function $W\colon \mathcal{Z}_{++} \times \mathcal{R} \to \mathcal{R}$ficontinuous and increasing in its second argumentfisuch thatfi for all $n, m \in \mathcal{Z}_{++}$fifor all $u \in \mathcal{R}^n$fiand for all $v \in \mathcal{R}^m$fi*

$$uRv \iff W(n, \Xi^n(u)) \geq W(m, \Xi^m(v)). \tag{12}$$

Proof. We define a value function with the desired properties. To do so, we employ a recursive construction.

First, we define $W^1\colon \mathcal{Z}^1 \times \mathcal{R} \to \mathcal{R}$ where $\mathcal{Z}^1 \subseteq \mathcal{Z}_{++}$ indexes the components of a vector $\mathbf{n}^1 = (n_j^1)_{j \in \mathcal{Z}^1}$ which is defined as follows. Let $n_1^1 = 1$. If there exists no $n \in \mathcal{Z}_{++} \setminus \{n_1^1\}$ such that $\mathcal{I}_{n_1^1}^n \neq \emptyset$, let $\mathbf{n}^1 = (n_1^1)$. If there is such a value of n, let

$$n_2^1 = \min\{n \in \mathcal{Z}_{++} \setminus \{n_1^1\} \mid \mathcal{I}_{n_1^1}^n \neq \emptyset\}. \tag{13}$$

Now suppose we have established $r-1$ components of the vector $\mathbf{n}^1$, where $r > 2$. If there exists no $n \in \mathcal{Z}_{++} \setminus \{n_1^1, \ldots, n_{r-1}^1\}$ such that $\cup_{j=1}^{r-1} \mathcal{I}_{n_j^1}^n \neq \emptyset$, let $\mathbf{n}^1 = (n_1^1, \ldots, n_{r-1}^1)$. If there is such a value of n, let

$$n_r^1 = \min\{n \in \mathcal{Z}_{++} \setminus \{n_1^1, \ldots, n_{r-1}^1\} \mid \cup_{j=1}^{r-1} \mathcal{I}_{n_j^1}^n \neq \emptyset\}. \tag{14}$$

This procedure generates a vector $\mathbf{n}^1 = (n_j^1)_{j \in \mathcal{Z}^1}$ with a finite or countable number of components. We now use this vector to construct the function W^1.

Step 1. Choose any a_1^1, $b_1^1 \in \mathcal{R}$ such that $a_1^1 < b_1^1$, and define a continuous and increasing function $W_{n_1^1}^1\colon \mathcal{R} \to \mathcal{R}$ such that $W_{n_1^1}^1(\mathcal{R}) = (a_1^1, b_1^1)$.

Step 2. If $\mathbf{n}^1 \neq (n_1^1)$, let $W_{n_2^1}^1(\xi) = W_{n_1^1}^1(\zeta)$ for all $\xi \in \mathcal{I}_{n_1^1}^{n_2^1}$ and $\zeta \in \mathcal{R}$ such that $(n_1^1, \zeta)\mathbf{I}(n_2^1, \xi)$. If $\mathcal{I}_{n_1^1}^{n_2^1}$ is bounded above, let $\xi_{\sup}^{n_2^1}$ be its least upper bound. Because $\mathcal{I}_{n_1^1}^{n_2^1}$ is open (Lemma 1), $\xi_{\sup}^{n_2^1} \notin \mathcal{I}_{n_1^1}^{n_2^1}$. Choose $b_2^1 \in \mathcal{R}$ such that $b_2^1 > b_1^1$ and extend $W_{n_2^1}^1$ to $[\xi_{\sup}^{n_2^1}, \infty)$ so that it is continuous and increasing and $W_{n_2^1}^1([\xi_{\sup}^{n_2^1}, \infty)) = [b_1^1, b_2^1)$. If $\mathcal{I}_{n_1^1}^{n_2^1}$ is not bounded above, let $b_2^1 = b_1^1$. If $\mathcal{I}_{n_1^1}^{n_2^1}$ is bounded below, let $\xi_{\inf}^{n_2^1}$ be its greatest lower bound and choose $a_2^1 \in \mathcal{R}$ such that $a_2^1 < a_1^1$ and extend $W_{n_2^1}^1$ to $(-\infty, \xi_{\inf}^{n_2^1}]$ so that it is continuous and increasing and $W_{n_2^1}^1((-\infty, \xi_{\inf}^{n_2^1}]) = (a_2^1, a_1^1]$. If $\mathcal{I}_{n_1^1}^{n_2^1}$ is not bounded below, let $a_2^1 = a_1^1$.

Step $r > 2$. Suppose the vector $\mathbf{n}^1$ contains at least $r > 2$ components and steps 1 to $r-1$ have been completed.

We first show that the set $\cup_{j=1}^{r-1}\mathcal{I}_{n_j^1}^{n_r^1}$ is connected. Suppose not. Then there exist $\hat{\jmath},\check{\jmath} \in \{1,\ldots,r-1\}$ and $\hat{\xi},\bar{\xi},\check{\xi} \in \mathcal{R}$ such that $\hat{\xi} > \bar{\xi} > \check{\xi}$, $\hat{\xi} \in \mathcal{I}_{n_{\hat{\jmath}}^1}^{n_r^1}$, $\check{\xi} \in \mathcal{I}_{n_{\check{\jmath}}^1}^{n_r^1}$, and $\bar{\xi} \notin \mathcal{I}_{n_j^1}^{n_r^1}$ for all $j \in \{1,\ldots,r-1\}$. Consequently,

$$(n_{\hat{\jmath}}^1,\xi)\mathbf{P}(n_r^1,\bar{\xi}) \text{ for all } \xi \in \mathcal{R}, \tag{15}$$

$$(n_r^1,\bar{\xi})\mathbf{P}(n_{\check{\jmath}}^1,\xi) \text{ for all } \xi \in \mathcal{R}, \tag{16}$$

and, for all $j \in \{1,\ldots,r-1\}$,

$$(n_j^1,\xi)\mathbf{P}(n_r^1,\bar{\xi}) \text{ for all } \xi \in \mathcal{R} \tag{17}$$

or

$$(n_r^1,\bar{\xi})\mathbf{P}(n_j^1,\xi) \text{ for all } \xi \in \mathcal{R}. \tag{18}$$

Let $\hat{S}$ be the set of all $j \in \{1,\ldots,r-1\}$ such that (17) is satisfied and $\check{S}$ be the set of all $j \in \{1,\ldots,r-1\}$ such that (18) is satisfied. (15) and (16) imply that both $\hat{S}$ and $\check{S}$ are nonempty, and (17) and (18) imply that $\{\hat{S},\check{S}\}$ is a partition of $\{1,\ldots,r-1\}$. Furthermore, it follows that, for all $j \in \hat{S}$ and for all $k \in \check{S}$,

$$(n_j^1,\xi)\mathbf{P}(n_k^1,\zeta) \text{ for all } \xi,\zeta \in \mathcal{R}. \tag{19}$$

Let $j^0 = \min\{j \in \hat{S}\}$ and $k^0 = \min\{k \in \check{S}\}$. Because $\{\hat{S},\check{S}\}$ is a partition of $\{1,\ldots,r-1\}$ and $\hat{S}$ and $\check{S}$ are nonempty, one of j^0 and k^0 is greater than one. Writing $s = \max\{j^0,k^0\}$, (19) implies that $\cup_{j=1}^{s-1}\mathcal{I}_{n_j^1}^{n_s^1} = \emptyset$, a contradiction. Therefore, $\cup_{j=1}^{r-1}\mathcal{I}_{n_j^1}^{n_r^1}$ is connected.

Let $W_{n_r^1}^1(\xi) = W_{n_j^1}^1(\zeta)$ for all $\xi \in \cup_{j=1}^{r-1}\mathcal{I}_{n_j^1}^{n_r^1}$ and ζ such that $(n_j^1,\zeta)\mathbf{I}(n_r^1,\xi)$. If $\cup_{j=1}^{r-1}\mathcal{I}_{n_j^1}^{n_r^1}$ is bounded above, let $\xi_{\text{sup}}^{n_r^1}$ be its least upper bound. Because $\cup_{j=1}^{r-1}\mathcal{I}_{n_j^1}^{n_r^1}$ is a union of open sets (Lemma 1), it is open and $\xi_{\text{sup}}^{n_r^1} \notin \cup_{j=1}^{r-1}\mathcal{I}_{n_j^1}^{n_r^1}$. Choose $b_r^1 \in \mathcal{R}$ such that $b_r^1 > b_{r-1}^1$ and extend $W_{n_r^1}^1$ to $[\xi_{\text{sup}}^{n_r^1},\infty)$ so that it is continuous and increasing and $W_{n_r^1}^1([\xi_{\text{sup}}^{n_r^1},\infty)) = [b_{r-1}^1,b_r^1)$. If $\cup_{j=1}^{r-1}\mathcal{I}_{n_j^1}^{n_r^1}$ is not bounded above, let $b_r^1 = b_{r-1}^1$. If $\cup_{j=1}^{r-1}\mathcal{I}_{n_j^1}^{n_r^1}$ is bounded below, let $\xi_{\text{inf}}^{n_r^1}$ be its greatest lower bound and choose $a_r^1 \in \mathcal{R}$ such that $a_r^1 < a_{r-1}^1$ and extend $W_{n_r^1}^1$ to $(-\infty,\xi_{\text{inf}}^{n_r^1}]$ so that it is continuous and increasing and $W_{n_r^1}^1((-\infty,\xi_{\text{inf}}^{n_r^1}]) = (a_r^1,a_{r-1}^1]$. If $\cup_{j=1}^{r-1}\mathcal{I}_{n_j^1}^{n_r^1}$ is not bounded below, let $a_r^1 = a_{r-1}^1$.

Because $\mathcal{Z}^1$ contains at most a countable number of elements, the above construction is well-defined. Let $W^1(n,\xi) = W_n^1(\xi)$ for all $(n,\xi) \in \mathcal{Z}^1 \times \mathcal{R}$.

If $\mathcal{Z}_{++} \setminus \mathcal{Z}^1 \neq \emptyset$, we define $W^2\colon \mathcal{Z}^2 \times \mathcal{R} \to \mathcal{R}$ where $\mathcal{Z}^2 \subseteq \mathcal{Z}_{++} \setminus \mathcal{Z}^1$ indexes the components of a vector $\mathbf{n}^2 = (n^2_j)_{j \in \mathcal{Z}^2}$ which is defined by letting $n^2_1 = \min \mathcal{Z}_{++} \setminus \mathcal{Z}^1$, and the remaining components of $\mathbf{n}^2$ (if any) are constructed recursively analogously to the construction of $\mathbf{n}^1$. The same steps as those used in the definition of W^1 can be applied to define W^2.

Let $t > 2$, and suppose we have defined the functions $W^1, \ldots, W^{t-1}$ in that fashion. If $\mathcal{Z}_{++} \setminus \left(\cup_{j=1}^{t-1} \mathcal{Z}^j\right) \neq \emptyset$, we define $W^t\colon \mathcal{Z}^t \times \mathcal{R} \to \mathcal{R}$ analogously. Because $\mathcal{Z}_{++}$ is countable, it follows that either there exists $T \in \mathcal{Z}_{++}$ such that $\mathcal{Z}_{++} = \cup_{t=1}^{T} \mathcal{Z}^t$ or $\mathcal{Z}_{++} = \cup_{t \in \mathcal{Z}_{++}} \mathcal{Z}^t$. In the first case, let $\mathcal{T} = \{1, \ldots, T\}$, and in the second case, let $\mathcal{T} = \mathcal{Z}_{++}$.

Now we define the function $W\colon \mathcal{Z}_{++} \times \mathcal{R} \to \mathcal{R}$ by letting, for all $t \in \mathcal{T}$, for all $n \in \mathcal{Z}^t$, and for all $\xi \in \mathcal{R}$,

$$W(n,\xi) = h^t(W^t(n,\xi)) \tag{20}$$

where each $h^t\colon \mathcal{R} \to \mathcal{R}$ is continuous and increasing, $h^t(\mathcal{R})$ is a nondegenerate and bounded open interval, and $h^t(\mathcal{R}) \cap h^s(\mathcal{R}) = \emptyset$ for all $s, t \in \mathcal{T}$ with $t \neq s$. By definition of the partition $\{\mathcal{Z}^t\}_{t \in \mathcal{T}}$ of $\mathcal{Z}_{++}$, if $n \in \mathcal{Z}^t$, $m \in \mathcal{Z}^s$, and $t \neq s$, either $(n,\xi)\mathbf{P}(m,\zeta)$ for all $\xi, \zeta \in \mathcal{R}$ or $(m,\zeta)\mathbf{P}(n,\xi)$ for all $\xi, \zeta \in \mathcal{R}$. Therefore, we can choose the functions h^t so that all rankings according to $\mathbf{R}$ are preserved by W and, thus, W represents $\mathbf{R}$. ■

The conditions of Theorem 2 are sufficient but not necessary for the existence of a value function. There are orderings violating extended continuity (but satisfying continuity and weak Pareto) that are representable but any representing value function is discontinuous in its second argument. An example of such an ordering is the one represented by

$$W(n,\xi) = \begin{cases} n\xi & \text{if } n = 1 \text{ and } \xi > 0, \\ n\big(\xi - 2\big) & \text{if } n \geq 2 \text{ and } \xi > 2, \\ \xi/2 - 1 & \text{if } n \geq 2 \text{ and } 0 < \xi \leq 2, \\ n\xi - 1 & \text{if } \xi \leq 0. \end{cases} \tag{21}$$

$W(1,\cdot)$ is discontinuous at $\xi = 0$ and the set $\{\zeta \in \mathcal{R} \mid (2,\zeta)\mathbf{R}(1,0)\} = (2,\infty) \cup \{0\}$ is not closed.

4. Concluding Remarks

Blackorby and Donaldson's [1984] result has been cited in various contributions to population ethics (see also Blackorby, Donaldson, and Weymark [1982] for an application in concentration measurement) and it is therefore important to present the above corrected version. It should be noted, however, that none of the characterization results we are aware of in this area are affected because the existence of a value function W is not assumed; rather, the characterizations proceed by working with the social-evaluation ordering R directly. In addition, all results assume only continuity and derive extended continuity from other axioms.

REFERENCES

Blackorby, C., W. Bossert, and D. Donaldson, 2000, Utilitarianism and the theory of justice, in *Handbook of Social fihoice and Welfarefi* K. Arrow, A. Sen, and K. Suzumura, eds., Elsevier, Amsterdam, forthcoming.

Blackorby, C. and D. Donaldson, 1984, Social criteria for evaluating population change, *Journal of Public Economics* **25**, 13–33.

Blackorby, C., D. Donaldson, and J.A. Weymark, 1982, A normative approach to industrial-performance evaluation and concentration indices, *European Economic Refi view* **19**, 89–121.