CAPITAL TAXATION IN A SIMPLE FINITE-HORIZON OLD MODEL

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Abstract

In a simple overlapping-generations model where the government has the power to levy commodity taxes and to implement inter–temporal transfers, we seek to characterise conditions under which capital taxation (or subsidization) does not form part of the optimal tax mix. It turns out that it can never be the case that captial taxes are identically zero along the Pareto frontier. Along the way, we derive and interpret the optimal tax formulae in the economy.

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> In a rational world peopled by honest citizens, you would not tax capital gains at all. To use a metaphor: we already tax the apples that fall from the tree, so why should we tax the tree itself?
>
> Edward Troup[1] in the *Financial Times* 13/14 November 1999

1. Introduction

The study of optimal taxation has been used to shed light on many problems in policy design. Among the great debates that tax theorists wade into is the question of whether capital income should be untaxed. Standard work in this area has concentrated on the issue of when capital income should form part of the direct income tax base. A main theme of this literature is that distortionary taxation of capital income is warranted whenever future consumption is complementary with leisure. In the overlapping generations context, Ordover and Phelps [1979] show that a zero marginal tax rate on savings income is a feature of an optimal nonlinear income tax when future consumption is separable from (current) labour supply in the direct utility function. This issue is also addressed in the arguments in favour of a consumption tax as opposed to an income tax. The putative advantage of a consumption tax is that before-tax and after-tax returns on savings are equal. This absence of distortion in the capital market is seen as a prerequisite for overall efficiency and the enhancement of economic growth. The support for this position is based, however, on very special models.[2] There seems to be some agreement that in the long run capital taxes should

[1] Tax partner at solicitors Simmons and Simmons and former Treasury adviser.

[2] See Auberebach [1994], Aurerbach and Kotlikoff [1987], Bradford [1987], Browning and Burbidge [1990], Kessleman [1994], and Meade [1978], for example.

go to zero.[3] These models range from steady-state analysis of OLG models to infinitely-lived representative agents. An in-depth survey of these results is available in Bernheim [1999].

In contrast to the above literature, we analyze a simple finite-horizon OLG model in which an affine income tax schedule and a full set of commodity taxes (or subsidies) are available as well as taxes on capital purchases and savings. We first characterize the Pareto-optimal tax structure; that is, the commodity, income, and capital/savings taxes that are efficient. We show that an extended version of the many person Ramsey rule holds.[4] While the issue of generational redistribution is not explicitly addressed, it is important to recall that different points on the Pareto frontier typically correspond to different distributions of income. We then address the issue of capital/savings taxation. In particular, we ask if there are any conditions on preferences and technologies such that capital/savings taxation should not be employed at a Pareto-optima. We conclude, somewhat surprisingly, that there are no restrictions such that capital taxes should be identically zero along the Pareto frontier. Optima without capital taxation may exist, but are not robust.

The model itself is simple. There are three periods: a start-up generation, and two generations that live for two periods each.[5] Both individuals — there is but one per generation — and the government can buy a storable good, capital, in order to transfer resources to the future. There is a competitive firm that uses capital, labour, and potentially other inputs, to produce a vector of outputs and the storable commodity. The government levies a one-hundred per cent profit tax on these firms. We allow the government to transfer abstract purchasing power to the old individual alive at time t, but we restrict these transfers to be

[3] See Chamley [1986], Diamond [1973], and Judd [1987,1999].

[4] See Diamond [1975]; a good textbook account is in Myles [1995]. Guesnerie [1995] uses an approach similar to the one we use here to derive optimal tax formulae in a static environment.

[5] There is nothing special about three; any finite horizon would do as long as the generations have lifetimes shorter than the horizon of the model.

of the same magnitude for all generations. This rules out the presence of lump–sum taxation, and makes our model a truly second–best one.[6] Thus, just as in the standard Ramsey optimal taxation problem, tax changes may have distributional consequences which cannot be neutralized by compensating adjustment in individual lump–sums. Capital taxes are no exception to this general rule.

Indeed, the theory of how changes in captial taxation affect the distribution of income is well–known, although this is usually expressed in terms of the distributions of gains and losses from a transition from an income tax to a consumption tax.[7] One interpretation of our results is that this type of redistribution is required to reach some points on the Pareto frontier. Two things must be true of the economy for this interpretation to have merit. First, redistribution of income across time must have a real impact on the economy. This is guaranteed by the existence of agents whose lifetimes are shorter than that of the planning horizon. Second, there must be a need to use "distortionary" redistributive devices. The restriction we impose on the power of lump–sum taxation is the simplest of a host of circumstances giving rise to this need.

The remainder of the paper is organized as follows. Section 2 outlines the model. This is followed by a tax reform analysis.[8] We start from an economy that is at an arbitrary equilibrium, and ask if there exist changes in taxes, transfers, and producer prices are strictly Pareto-improving and equilibrium preserving. By describing the feasible, Pareto–improving directions of policy reform, we also furnish a characterization of the (local) second–best Pareto optima. Our main result is presented and interpreted in Section 5. We offer some concluding remarks in Section 6, and collect many of the mathematical details in an appendix.

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[6] A similar situation would hold if we allowed many individuals per generation, unrestricted transfers across generations, but a common transfer within generations.

[7] See Boadway and Wildsain [1994] for an exposition of this issue.

[8] See Guesnerie [1977,1995], Diewert [1978], Weymark [1979].

2. The Model

We consider the simplest possible overlapping generations model.[9] The economy lasts for three periods: a start-up phase, a single period of the type usually examined in overlapping generations models, and a shut-down period.

2.1. Goods and Consumers

There is a single consumer in each generation, so consumer and generation are used interchangeably. Consumers have preferences over a vector $\alpha \in \mathcal{R}^n$ of nonstorable goods and services and a single storable good, κ. The latter is the sole means of transferring resources across time.

An initial generation, denoted by 0, is born old. It enters at date 1. It consumes goods and services, α_1^0 and receives common lump-sum transfer, m. Also alive at time 1 is a generation born young. This generation lives for two periods. During period 1, it consumes α_1^1 and may also purchase an amount of the storable good, κ_1^1, to carry forward with it into the second period. In period 2 it spends its accumulated wealth and its lump-sum transfer, m, on the consumption of α_2^1. A second young generation is born in period 2; it works, consumes, and saves. In period 3, the final period of our model, this generation sells its capital stock, κ_2^2, receives its lump-sum transfer,m, and consumes α_3^3.[10]

The production sector is composed of an aggregate profit–maximizing firm whose technology may change over time. During periods 1 and 2, this firm can produce non–durables

[9] This is a simple version of the model introduced by Allais [1947], Samuelson [1958], and analyzed by Diamond [1965]. The model is described in full detail in Blackorby and Brett [1999]

[10] Negative elements of α_t^τ are interpreted as the supply of factor services, such as labour.

in amounts a and the storable good in amount b, using a and k as inputs. In period 3 it does not produce any b.[11]

The array of prices in this economy is very large, comprised of a set consumer prices and a set of producer prices for each commodity at each date in time. Let p_t be the producer price vector for a_t.[12] π_t denotes the corresponding consumer price vector. r_t is the producer price of the storable good at time t, while ρ_t is its consumer price. In addition the firm buys at time t the capital stock from generation $t-1$ at a price s_{t-1} while generation $t-1$ receives σ_{t-1}. Taxes are defined implicitly by

$$\pi_t = p_t + \tau_t^a, \quad \sigma_t = s_t + \tau_t^k, \quad \text{and} \quad \rho_t = r_t + \tau_t^b. \tag{2.1}$$

We assume that the government may bestow a common lump–sum income transfer upon each generation. Specifically, at date t it transfers m to the old generation at that time. These transfers are financed by a combination of commodity taxes, a one–hundred per cent pure profit tax,[13] and purchases of the storable good. We denote these purchases by κ_1^g in period 1 and κ_2^g in period 2. We note in passing that the government's redistribution of income across time is mediated by the capital market.

Given the prices prevailing in period 1, and its lump-sum transfer, the budget constraint of generation 0 is[14]

$$\pi_1^T \alpha_1^0 \leq m. \tag{2.2}$$

Its preferences are represented by an indirect utility function given by

$$u_0 = V^0(\pi_1, m). \tag{2.3}$$

[11] We use roman letters to indicate quantities produced and greek letters to indicate quantities consumed. That is, the symbols α and a refer to goods of identical characteristics. The same correspondence applies to κ and k. An inconsistency in notation arises in that the supply of κ is denoted b.

[12] We express all prices in present value form.

[13] Alternatively, one could assume constant returns-to-scale, implying zero profits.

[14] We use subscripts to denote the date at which a commodity is produced or consumed. When ambiguity is possible, we use superscripts to denote the birth date of the consuming agent.

Generation 1 faces an inter–temporal decision problem. It allocates its period 1 service income among goods and services consumed in that period and purchases of the storable good. At the beginning of the second period, it sells its capital to the firm, receiving σ_1 per unit. It combines this with its lump–sum payment from the government to finance (net) transactions in period 2. Thus, its behaviour is consistent with the joint budget constraints:

$$\begin{aligned} \pi_1^T \alpha_1^1 + \rho_1 \kappa_1^1 \leq 0; \\ \pi_2^T \alpha_2^1 \leq \sigma_1 \kappa_1^1 + m. \end{aligned} \tag{2.4}$$

If generation 1 has positive savings,[15] its behaviour is also consistent with the single budget constraint

$$\frac{\sigma_1}{\rho_1} \pi_1^T \alpha_1^1 + \pi_2^T \alpha_2^1 \leq m. \tag{2.5}$$

When this is the case, its indirect utility function is given by

$$u_1 = V^1(\tilde{\pi}_1, \pi_2, m), \tag{2.6}$$

where

$$\tilde{\pi}_1 := \frac{\sigma_1}{\rho_1} \pi_1. \tag{2.7}$$

Similarly, the value function of generation 2—conditional on positive savings—is given by

$$u_2 = V^2(\tilde{\pi}_2, \pi_3, m). \tag{2.8}$$

We assume that the preferences are such that the indirect utility functions are differentially strongly quasi-convex.[16]

In each period, the firm uses the capital it purchases from the old in combination with the services supplied by the two generations alive at that time to produce a vector of (net) outputs. We assume that the within–period profit functions of the firm are twice continuously

[15] This is the only case that we analyze because we are interested in capital/savings taxation; in this case, it is clear from (2.4) that at least one element of α_1^1 must be negative — perhaps labour.

[16] See Blackorby and Diewert [1979].

differentiable and strongly convex in each period.[17] This means that we may describe net supplies by differentiable functions

$$a_1 = A^1(p_1, r_1), \quad b_1 = B^1(p_1, r_1), \quad \text{and} \quad k_1 = K^1(p_1, r_1), \tag{2.9}$$

$$a_2 = A^2(s_1, p_2, r_2), \quad b_2 = B^2(s_1, p_2, r_2), \quad \text{and} \quad k_2 = K^2(s_1, p_2, r_2), \tag{2.10}$$

and

$$a_3 = A^3(s_2, p_3), \quad \text{and} \quad k_3 = K^3(s_2, p_3). \tag{2.11}$$

We have suppressed the level of the accumulated capital stock in the supply functions as a matter of notation. Because the technology is not assumed to be the same in each period, this formulation is consistent with any rate of capital depreciation.

A collection of prices give rise to an equilibrium if

$$\begin{aligned}
-\alpha_1^0 - \alpha_1^1 + a_1 &\geq 0, \\
-\kappa_1^1 - \kappa_1^g + b_1 &\geq 0, \\
\kappa_1^1 + \kappa_1^g - k_2 &\geq 0, \\
-\alpha_2^1 - \alpha_2^2 + a_2 &\geq 0, \\
-\kappa_2^2 - \kappa_2^g + b_2 &\geq 0, \\
\kappa_2^2 + \kappa_2^g - k_3 &\geq 0, \\
-\alpha_3^2 + a_3 &\geq 0, \\
\kappa_1^g &\geq 0, \quad \text{and} \\
\kappa_2^g &\geq 0.
\end{aligned} \tag{2.12}$$

That is, all markets— for both storable and non–storable commodities — clear. The capital market clearing conditions include the demand for capital purchases by the government. Viewing these purchases as a form of government expenditure, one can show that the government budget is balanced in each period.[18]

[17] See Diewert, Avriel, and Zang [1981].

[18] See Chapter 2 in Guesnerie [1995] for a general discussion of this issue; a proof for the model outlined here is given in Blackorby and Brett [1999].

3. Feasible Pareto Improving Policy Reforms

We assume that the taxation authority has control over lump–sum transfers, commodity taxes and its capital purchases. Given our assumptions on technology, we may also treat producer prices as under the control of the planner. In fact, producer prices adjust to changes in taxes, but these adjustments are captured by the equilibrium conditions. We wish to investigate if, starting at an initial tight equilibrium[19] with positive saving, there exist directions of policy reform that are strictly Pareto improving and equilibrium preserving. If no such directions exist, then the economy is at a local second best optimum.

We denote a direction of policy reform by γ, where

$$\gamma^T := [\gamma_p^T, \gamma_\tau^T, \gamma_m^T, \gamma_\kappa^T] \tag{3.1}$$

and

$$\begin{aligned}
\gamma_p^T &:= [dp_1^T, dr_1, ds_1, dp_2^T, dr_2, ds_2, dp_3^T];\\
\gamma_\tau^T &:= [d\tau_1^{aT}, d\tau_1^b, d\tau_1^k, d\tau_2^{aT}, d\tau_2^b, d\tau_2^k, d\tau_3^{aT}];\\
\gamma_m^T &:= [dm];\\
\gamma_\kappa^T &:= [d\kappa_1^g, d\kappa_2^g].
\end{aligned} \tag{3.2}$$

The partition of γ corresponds to changes in producer prices, taxes, lump-sum transfers, and government capital purchases.

Given our assumption of positive savings in each period, we may use the indirect utility functions to characterize changes in consumer prices and lump-sum transfers that improve the well-being of each generation. Using Roy's theorem and the envelope conditions these can be written as

$$du_0 > 0 \leftrightarrow - \quad \alpha_1^{0T} d\pi_1 + dm > 0, \tag{3.3}$$

$$du_1 > 0 \leftrightarrow - \quad \frac{\sigma_1}{\rho_1}\alpha_1^{1T} d\pi_1 + \frac{\sigma_1}{\rho_1^2}\pi_1^T \alpha_1^1 d\rho_1 - \frac{1}{\rho_1}\pi_1^T \alpha_1^1 d\sigma_1 - \alpha_2^{1T} d\pi_2 + dm > 0 \tag{3.4}$$

[19] An equilibrium is said to be tight if all relations in (2.12) (except possibly the last two) hold with equality.

and

$$du_2 > 0 \leftrightarrow - \quad \frac{\sigma_2}{\rho_2}\alpha_2^{2T} d\pi_2 + \frac{\sigma_2}{\rho_2^2}\pi_2^T \alpha_2^2 d\rho_2 - \frac{1}{\rho_2}\pi_2^T \alpha_2^2 d\sigma_2 - \alpha_3^{2T} d\pi_3 + dm > 0. \tag{3.5}$$

For future reference, we decompose the left hand sides of (3.3)-(3.5) into consumer price effects and income effects. Collecting coeficients in matrix form gives

$$P_\pi := \begin{bmatrix} -\alpha_1^{0T} & 0 & 0 & 0_n^T & 0 & 0 & 0_n^T \\ -\frac{\sigma_1}{\rho_1}\alpha_1^{1T} & \frac{\sigma_1}{\rho_1^2}\pi_1^T\alpha_1^1 & -\frac{1}{\rho_1}\pi_1^T\alpha_1^1 & -\alpha_2^{1T} & 0 & 0 & 0_n^T \\ 0_n^T & 0 & 0 & -\frac{\sigma_2}{\rho_2}\alpha_2^{2T} & \frac{\sigma_2}{\rho_2^2}\pi_2^T\alpha_2^2 & -\frac{1}{\rho_2}\pi_2^T\alpha_2^2 & -\alpha_3^{2T} \end{bmatrix} \tag{3.6}$$

and

$$P_m := \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}. \tag{3.7}$$

In order to describe the feasible directions of policy reform, we define a collection of matrices.[20] First,

$$E_\pi := \begin{bmatrix} -\nabla_{\pi_1}\alpha_1 & -\nabla_{\rho_1}\alpha_1^1 & -\nabla_{\sigma_1}\alpha_1^1 & -\nabla_{\pi_2}\alpha_1^1 & 0_n & 0_n & 0_{n\times n} \\ -\nabla_{\pi_1}\kappa_1^1 & -\nabla_{\rho_1}\kappa_1^1 & -\nabla_{\sigma_1}\kappa_1^1 & -\nabla_{\pi_2}\kappa_1^1 & 0 & 0 & 0_n^T \\ +\nabla_{\pi_1}\kappa_1^1 & +\nabla_{\rho_1}\kappa_1^1 & +\nabla_{\sigma_1}\kappa_1^1 & +\nabla_{\pi_2}\kappa_1^1 & 0 & 0 & 0_n^T \\ -\nabla_{\pi_1}\alpha_2^1 & -\nabla_{\rho_1}\alpha_2^1 & -\nabla_{\sigma_1}\alpha_2^1 & -\nabla_{\pi_2}\alpha_2 & -\nabla_{\rho_2}\alpha_2^2 & -\nabla_{\sigma_2}\alpha_2^2 & -\nabla_{\pi_3}\alpha_2^2 \\ 0_n^T & 0 & 0 & -\nabla_{\pi_2}\kappa_2^2 & -\nabla_{\rho_2}\kappa_2^2 & -\nabla_{\sigma_2}\kappa_2^2 & -\nabla_{\pi_3}\kappa_2^2 \\ 0_n^T & 0 & 0 & +\nabla_{\pi_2}\kappa_2^2 & +\nabla_{\rho_2}\kappa_2^2 & +\nabla_{\sigma_2}\kappa_2^2 & +\nabla_{\pi_3}\kappa_2^2 \\ 0_{n\times n} & 0_n & 0_n & -\nabla_{\pi_2}\alpha_3^2 & -\nabla_{\rho_2}\alpha_3^2 & -\nabla_{\sigma_2}\alpha_3^2 & -\nabla_{\pi_3}\alpha_3^2 \end{bmatrix}. \tag{3.8}$$

Each row of (3.8) corresponds to a relation in (2.12), with elements corresponding to the change in the left–hand side in (2.12) due to an infinitesimal change in a consumer price.

[20] Variables without superscripts denote within–period aggregate demands.

To deal with the demogrant, producer price changes and government capital purchases, we introduce:

$$E_m := \begin{bmatrix} -\nabla_m \alpha_1^0 - \nabla_m \alpha_1^1 \\ -\nabla_m \kappa_1^1 \\ \nabla_m \kappa_1^1 \\ -\nabla_m \alpha_2^1 - \nabla_{m_3} \alpha_2^2 \\ -\nabla_m \kappa_2^2 \\ \nabla_m \kappa_2^2 \\ -\nabla_m \alpha_3^2 \end{bmatrix}; \tag{3.9}$$

$$E_p := \begin{bmatrix} \nabla_{p_1} a_1 & \nabla_{r_1} a_1 & 0_n & 0_{n\times n} & 0_n & 0_n & 0_{n\times n} \\ \nabla_{p_1} b_1 & \nabla_{r_1} b_1 & 0 & 0_n^T & 0 & 0 & 0_n^T \\ 0_n^T & 0 & -\nabla_{s_1} k_2 & -\nabla_{p_2} k_2 & -\nabla_{r_2} k_2 & 0 & 0_n^T \\ 0_{n\times n} & 0_n & \nabla_{s_1} a_2 & \nabla_{p_2} a_2 & \nabla_{r_2} a_2 & 0 & 0_n^T \\ 0_n^T & 0 & \nabla_{s_1} b_2 & \nabla_{p_2} b_2 & \nabla_{r_2} b_2 & 0 & 0_n^T \\ 0_n^T & 0 & 0 & 0_n^T & 0 & -\nabla_{s_2} k_3 & -\nabla_{p_3} k_3 \\ 0_{n\times n} & 0_n & 0_n & 0_{n\times n} & 0_n & \nabla_{s_2} a_3 & \nabla_{p_3} a_3 \end{bmatrix}; \tag{3.10}$$

and

$$E_\kappa := \begin{bmatrix} 0 & 0 \\ -1 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & -1 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}. \tag{3.11}$$

In addition to the above we have the nonnegativity constraints on the government's capital purchases. For a direction of reform to be feasible, it must be the case that the after–reform level of capital purchases must be nonnegative as well. This is summarized by the relations

$$d\kappa_1^g + \kappa_1^g \geq 0 \quad \text{and} \quad d\kappa_2^g + \kappa_2^g \geq 0. \tag{3.12}$$

Consumer utility depends on income and consumer prices alone. The latter are the sum of producer prices and taxes. Employing the notation of the previous subsection, a set of changes is strictly Pareto-improving if and only if

$$P_\pi \gamma_p + P_\pi \gamma_\tau + P_m \gamma_m + \mathbf{0}\gamma_\kappa \gg 0 \tag{3.13}$$

where $\mathbf{0}$ is an appropriately dimensioned matrix of zeros. A direction is equilibrium-preserving if and only if

$$[E_\pi + E_p]\gamma_p + E_\pi \gamma_\tau + E_m \gamma_m + E_\kappa \gamma_\kappa \geq 0. \tag{3.14}$$

In addition the government capital constraints, (3.12), must be satisfied. There are strict Pareto-improving changes that are simultaneously equilibrium- preserving if and only (3.13), (3.14), and (3.12) have a solution. Together these constitute a non homogeneous system of linear equalities and inequalities. It is straightforward to show that this system is equivalent to the following homogeneous system

$$\begin{bmatrix} P_\pi & P_\pi & P_m & 0_{n\times 2} & 0_n \\ 0_n^T & 0_n^T & 0_3^T & 0_2^T & 1 \end{bmatrix} \begin{bmatrix} \gamma \\ \gamma_\eta \end{bmatrix} \gg 0 \tag{3.15}$$

and

$$\begin{bmatrix} E_\pi + E_p & E_\pi & E_m & E_\kappa & 0 \\ 0_{2\times n} & 0_{2\times n} & 0_{2\times 3} & I_{2\times 2} & \kappa^g \end{bmatrix} \begin{bmatrix} \gamma \\ \gamma_\eta \end{bmatrix} \geq 0 \tag{3.16}$$

where γ_η is a place–holding variable.[21] If there is no solution to (3.16) we are at a second-best optimum. Using Motzkin's Theorem[22] the economy is at a second-best optimum if and only if

$$\begin{bmatrix} \xi^T & \theta \end{bmatrix} \begin{bmatrix} [P_\pi & P_\pi & P_m & \mathbf{0} & 0 \\ 0_n^T & 0_n^T & 0_3^T & 0_2^T & 1 \end{bmatrix} + \begin{bmatrix} v^T & \eta^T \end{bmatrix} \begin{bmatrix} E_\pi + E_p & E_\pi & E_m & E_\kappa & 0 \\ 0_{2\times n} & 0_{2\times n} & 0_{2\times 3} & I_{2\times 2} & \kappa^g \end{bmatrix} = 0, \tag{3.17}$$

for some collection of multipliers such that $0 \neq [\xi^T, \theta] \geq 0^T$ and $[v^T, \eta^T] \geq 0^T$.

[21] For a proof of this, see the appendix to Blackorby and Brett [1999].

[22] See Mangasarian [1969, pp. 28-29] for a statement and proof of this result.

4. Pareto-efficient Taxation

The relations (3.17) contain the first–order conditions for a Pareto optimum, and implicitly define the efficient taxes and demogrant. The multipliers correspond to shadow values. Specifically, the vector v contains the social marginal values of commodities, and ξ is the vector of social marginal values of the incomes of the various generations (consumers). Several easy–to–interpret optimal tax rules can be derived. For instance, the third column of (3.17) implies

$$\xi^T P_m + v^T E_m = 0, \tag{4.1}$$

or, upon expansion,

$$\begin{aligned}\xi_1 + \xi_2 + \xi_3 = {} & v_2^T(\nabla_m \alpha_1^0 + \nabla_m \alpha_1^1) + (v_4 - v_3)\nabla_m \kappa_1^1 \\ & + v_5^T(\nabla_m \alpha_2^1 + \nabla_m \alpha_2^2) + (v_7 - v_8)\nabla_m \kappa_2^2 + v_8^T \nabla_m \alpha_3^2.\end{aligned} \tag{4.2}$$

The left–hand side of (4.2) is the marginal value of the demogrant, comprised of the sum of the social marginal values of income. The right hand side is the cost, measured at shadow prices, of meeting the changes in demand induced by a marginal change in the demogrant. These two quantities must be equal at a local second–best optimum.

Subtracting the second column of (3.17) from the first yields

$$v^T E_p = 0. \tag{4.3}$$

From (4.3) and the strong convexity of the firm's profit functions, it follows that[23]

$$v^T = \Big[\mu_1(p_1^T, r_1), \mu_2(s_1, p_2^T, r_2), \mu_3(s_2, p_3^T)\Big] \tag{4.4}$$

where μ_t is a function only of period t producer prices. That is, shadow prices in the economy are proportional to producer prices in each period. No such claim can be made directly for intertemporal shadow prices, which depend upon the values of μ_t.

[23] From a similar argument in Section 5 of Blackorby and Brett [1999].

The fifth column of (3.17) has no analogue in the static Ramsey problem. Expanding it yields

$$\theta + \eta_1 \kappa_1^g + \eta_2 \kappa_2^g = 0. \tag{4.5}$$

But each term in (4.5) is nonnegative. Hence, they are all zero and we arrive at

$$\eta_1 \kappa_1^g = 0 \quad \text{and} \quad \eta_2 \kappa_2^g = 0, \tag{4.6}$$

the standard complementary slackness conditions associated with the constraints on government capital purchases. These conditions partition the set of Pareto optima into four regions of interest, depending upon the timing of government saving. The region of $\kappa_1^g = 0$ and $\eta_1 > 0$ corresponds to a situation where the government would like to transfer more resources into earlier periods, but is prevented from doing so. The region in which $\eta_1 = 0$ and $\eta_2 = 0$ corresponds to a case when the planner is not capital constrained and is saving in both periods. The fourth column of (3.17) implies

$$v^T E_\kappa + \eta^T = 0. \tag{4.7}$$

Thus, when the government saves in both periods, (4.7) implies

$$\mu_{t+1} s_t = \mu_t r_t \tag{4.8}$$

for $t = 1, 2$. That is, the ratio of intertemporal shadow prices is equal to the ratio of the price that firms must pay for the stored good at the beginning of period t to what it can sell the storable good for at the end of the period.

The second column of (3.17) incorporates the information commonly contained in Ramsey formulae. Specifically,

$$\xi^T P_\pi + v^T E_\pi = 0. \tag{4.9}$$

Using an argument from the appendix, (4.9) and (4.4) yield

$$\begin{aligned} \xi_1 + \xi_2 + \xi_3 =& \mu_1 p_1^T \nabla_m \alpha_1^0 + \mu_1 p_1^T \nabla_m \alpha_1 + \mu_2 p_2^T \nabla_m \alpha_2 + \mu_3 p_3^T \nabla_m \alpha_3^2 \\ & - (\mu_2 s_1 - \mu_1 r_1) \nabla_m \kappa_1^1 - (\mu_3 s_2 - \mu_2 r_2) \nabla_m \kappa_2^2, \end{aligned} \tag{4.10}$$

$$\left[\, \xi_1 \alpha_1^{0T} + \xi_2 \tfrac{\sigma_1}{\rho_1} \alpha_1^{1T} \quad \xi_2 \alpha_2^{1T} + \xi_3 \tfrac{\sigma_2}{\rho_2} \alpha_2^{2T} \quad \xi_3 \alpha_3^{2T} \,\right]$$

$$= - \left[\, \mu_1 p_1^T \quad \mu_2 p_2^T \quad \mu_3 p_3^T \,\right] \begin{bmatrix} \nabla_{\pi_1} \alpha_1 & \nabla_{\pi_2} \alpha_1^1 & \mathbf{0} \\ \nabla_{\pi_1} \alpha_2^1 & \nabla_{\pi_2} \alpha_2 & \nabla_{\pi_3} \alpha_2^2 \\ \mathbf{0} & \nabla_{\pi_2} \alpha_3^2 & \nabla_{\pi_3} \alpha_3^2 \end{bmatrix}$$

$$+ \left[\, (\mu_2 s_1 - \mu_1 r_1) \nabla_{\pi_1} \kappa_1^1 \quad (\mu_2 s_1 - \mu_1 r_1) \nabla_{\pi_2} \kappa_1^1 + (\mu_3 s_2 - \mu_2 r_2) \nabla_{\pi_2} \kappa_2^2 \quad (\mu_3 s_2 - \mu_2 r_2) \nabla_{\pi_3} \kappa_2^2 \,\right] \tag{4.11}$$

$$-\xi_2 \rho_1 \kappa_1^1 = -\mu_1 p_1^T \nabla_{\tilde{\pi}_1} \alpha_1^1 \pi_1 - \mu_2 p_2^T \nabla_{\tilde{\pi}_1} \alpha_2^1 \pi_1 + (\mu_2 \sigma_1 - \mu_2 r_1) \nabla_{\tilde{\pi}_1} \kappa_1^1 \pi_1 \tag{4.12}$$

and

$$-\xi_3 \rho_2 \kappa_2^2 = -\mu_2 p_2^T \nabla_{\tilde{\pi}_2} \alpha_2^2 \pi_2 - \mu_3 p_3^T \nabla_{\tilde{\pi}_2} \alpha_3^2 \pi_2 + (\mu_3 s_2 - \mu_2 r_2) \nabla_{\tilde{\pi}_2} \kappa_2^2 \pi_2 \tag{4.13}$$

When (4.8) holds, the last line of (4.11) is zero and what we are left with is close to the standard Mirrlees' result.[24] That is, the social value of the demands and supplies is equal to the cost of the change in commodity and service taxes valued at producer prices.

Using the Slutsky equation for each generation allows one to rewrite this to obtain the many person Ramsey rule. From (4.12) and (4.13) we obtain a result relating the value of savings of generation one to changes in the consumer cost of changes in consumption again valued at producer prices. This is the most unusual result. Its literal meaning is that the social value of the savings of generation one must be equal to the value, at consumer prices, of the induced change in the demand for good and services times the shadow prices of these commodities and services. To interpret this result it is useful to consider the effect of giving generation one a voucher that it can spend on capital purchases only. The social value of

[24] See Mirrlees [1986] and Guesnerie [1995].

this would be minus the left–hand side of (4.12). Generation one would spend this windfall on goods and services, perhaps some in each period of its life. The first two terms of (4.12) account for the costs, at the social shadow prices prevailing in the period of consumption, of these changes. From (4.13) we obtain a similar result for the value of the savings of generation two.

In the other three regions we obtain results that are quite different. The index of discouragement in consumption must be adjusted for the induced changes in savings which are valued at the differences in the social value of capital in two adjacent time periods, $\mu_{t+1}s_1 - \mu_t r_t$. This is the result of the fact that in these regions the government is moving into regions of the Pareto-set where it wishes that it could move capital from the future to the present. This in turn entails that the shadow value of capital is different in the two periods.

5. Capital Taxation

We are now in a position to pose the principal question of the paper; what conditions on preferences and technologies are necessary and sufficient to imply that capital taxation is unnecessary for an efficient outcome. It is well-known that in certain classes of models with sufficient structure capital taxation is redundant.[25] The model set out above is ideal to test this intuition. Each generation has a lifetime that is shorter than that of the economy. Commodity and service taxes are an essential part of any Pareto-efficient outcome because the government does not have access to individual lump-sum transfers, but only a uniform lump-sum transfer—a demogrant. Given that one of the components of α_t^t must be negative, this is admits the case of an anonymous affine income-tax schedule with linear taxes for all other commodities.

[25] See Auerbach and Hines [1999] for a discussion and for references.

In the appendix it is shown that there are six harmless normalizations in the model,

$$p_{11} = 1, \quad s_1 = 1, \quad s_2 = 1, \quad \tau^a_{11} = 0, \quad \tau^k_1 = 0, \quad \text{and} \quad \tau^k_2 = 0. \tag{5.1}$$

That is, we can normalize one producer price in each period, either the tax on capital or the tax on savings in each period, but not both, and one consumer price (not per period). We have chosen arbitrarily to set the producer price of capital in each period equal to one and the tax on savings in each period and the initial capital stock equal to zero.

To pose the question, we ask under what circumstances can we—without loss of generality—set the tax on the purchases of the storable good equal to zero in addition to the normalizations given above. That is, under what conditions are efficient equilibria consistent with (5.1) and

$$\tau^b_1 = 0 \quad \text{and} \quad \tau^b_2 = 0. \tag{5.2}$$

To phrase this question we employ the matrices

$$\mathcal{I} = \begin{bmatrix} 1, 0^T_n & 0 & 0 & 0^T_n & 0 & 0 & 0^T_n \\ 0^T_n & 0 & 1 & 0^T_n & 0 & 0 & 0^T_n \\ 0^T_n & 0 & 0 & 0^T_n & 0 & 1 & 0^T_n \end{bmatrix} \tag{5.3}$$

$$\hat{\mathcal{I}} = \begin{bmatrix} 1, 0^T_{n-1} & 0 & 0 & 0^T_n & 0 & 0 & 0^T_n \\ 0^T_n & 1 & 0 & 0^T_n & 0 & 0 & 0^T_n \\ 0^T_n & 0 & 1 & 0^T_n & 0 & 0 & 0^T_n \\ 0^T_n & 0 & 0 & 0^T_n & 1 & 0 & 0^T_n \\ 0^T_n & 0 & 0 & 0^T_n & 0 & 1 & 0^T_n \end{bmatrix}. \tag{5.4}$$

The six normalizations plus (5.2) are imposed by

$$\begin{bmatrix} \mathcal{I} \\ \mathbf{0} \end{bmatrix} \gamma_p + \begin{bmatrix} \mathbf{0} \\ \hat{\mathcal{I}} \end{bmatrix} \gamma_\tau + \begin{bmatrix} \mathbf{0} \\ \mathbf{0} \end{bmatrix} \gamma_m + \begin{bmatrix} \mathbf{0} \\ \mathbf{0} \end{bmatrix} \gamma_\kappa + \begin{bmatrix} \mathbf{0} \\ \mathbf{0} \end{bmatrix} \gamma_\eta = 0. \tag{5.5}$$

Using Motzkin's Theorem the economy is at a second-best optimum if and only if

$$\begin{bmatrix} \xi^T & \theta \end{bmatrix} \begin{bmatrix} [P_\pi & P_\pi & P_m & \mathbf{0} & 0 \\ 0^T_n & 0^T_n & 0 & 0^T_2 & 1 \end{bmatrix} + \begin{bmatrix} v^T & \eta^T \end{bmatrix} \begin{bmatrix} E_\pi + E_p & E_\pi & E_m & E_\kappa & 0 \\ 0_{2\times n} & 0_{2\times n} & 0_2 & I_{2\times 2} & \kappa^g \end{bmatrix} \tag{5.6}$$

$$+ (w^T, z^T) \begin{bmatrix} \mathcal{I} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \hat{\mathcal{I}} & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix} = 0$$

where $0 \neq [\xi^T, \theta] \geq 0^T$ and $[v^T, \eta^T] \geq 0^T$.

Expanding (5.6) yields

$$\xi^T P_\pi + v^T(E_\pi + E_p) + w^T \mathcal{I} = 0, \tag{5.7}$$

$$\xi^T P_\pi + v^T E_\pi + z^T \hat{\mathcal{I}} = 0, \tag{5.8}$$

$$\xi^T P_m + v^T E_m = 0, \tag{5.9}$$

$$v^T E_\kappa + \eta^T = 0, \tag{5.10}$$

$$\theta + \eta^T \kappa^g = 0. \tag{5.11}$$

In order for capital taxation to be redundant, it must be that w and z are both identically zero. However, if z is not zero, then w cannot be. Therefore a necessary condition for the redundancy of capital taxation is that z be equal to zero. Expanding (5.8) yields

$$\sum_t \xi_t - v_2^T \nabla_m \alpha_1 + (v_4 - v_3)\nabla_m \kappa_1^1 - v_5^T \nabla_m \alpha_2 + (v_7 - v_6)\nabla_m \kappa_2^2 - v_8^T \nabla_m \alpha_3^2 = 0, \tag{5.12}$$

$$-\xi_1 \alpha_1^{0T} - \xi_2 \alpha_1^{1T} - v^T \nabla_{\pi_1} \alpha_1 + (v_4 - v_3)\nabla_{\pi_1} \kappa_1^1 - v_5^T \nabla_{\pi_1} \alpha_2^1 + (z_1, 0_{n-1}^T) = 0, \tag{5.13}$$

$$\xi_2 \frac{\sigma_1}{\rho_1^2} \pi_1^T \alpha_1^1 - v_2^T \nabla_{\rho_1} \alpha_1^1 + (v_4 - v_3)\nabla_{\rho_1} \kappa_1^1 - v_5^T \nabla_{\rho_1} \alpha_2^1 + z_2 = 0, \tag{5.14}$$

$$-\xi_2 \frac{1}{\rho_1} \pi_1^T \alpha_1^1 - v_2^T \nabla_{\sigma_1} \alpha_1^1 + (v_4 - v_3)\nabla_{\sigma_1} \kappa_1^1 - v_5^T \nabla_{\sigma_1} \alpha_2^1 + z_3 = 0, \tag{5.15}$$

$$-\xi_2 \alpha_2^{1T} - \xi_3 \frac{\sigma_2}{\rho_2} \alpha_2^{2T} - v_2^T \nabla_{\pi_2} \alpha_1^1 + (v_4 - v_3)\nabla_{\pi_2} \kappa_1^1 - v_5^T \nabla_{\pi_2} \alpha_2 + (v_7 - v_6)\nabla_{\pi_2} \kappa_2^2 - v_8^T \nabla_{\pi_2} \alpha_3^2 = 0, \tag{5.16}$$

$$\xi_3 \frac{\sigma_2}{\rho_2^2} \pi_2^T \alpha_2^2 - v_5^T \nabla_{\rho_2} \alpha_2^2 + (v_7 - v_6)\nabla_{\rho_2} \kappa_2^2 - v_8^T \nabla_{\rho_2} \alpha_2^2 + z_4 = 0, \tag{5.17}$$

$$-\xi_3 \frac{1}{\rho_2} \pi_2^T \alpha_2^2 - v_5^T \nabla_{\sigma_2} \alpha_2^2 + (v_7 - v_6)\nabla_{\sigma_2} \kappa_2^2 - v_8^T \nabla_{\sigma_2} \alpha_2^2 + z_5 = 0, \tag{5.18}$$

and

$$-\xi_3 \alpha_3^{2T} - v_5^T \nabla_{\pi_3} \alpha_2^2 + (v_7 - v_6)\nabla_{\pi_3} \kappa_2^2 - v_8^T \nabla_{\pi_3} \alpha_3^2 = 0. \tag{5.19}$$

From the argument in the appendix, z_1 is equal to zero.[26] To find the conditions under which z_2 and z_3 are zero, multiply (5.14) by ρ_1, (5.15) by σ_1, and add. Using the homogeneity of the demand functions yields

$$z_2\rho_1 + z_3\sigma_1 = 0. \tag{5.20}$$

Hence we cannot in general normalize both the capital tax and savings tax to be zero. In fact z_2 and z_3 are identically zero if and only if

$$\xi_2\frac{1}{\rho_1}\pi_1^T\alpha_1^1 - v_2^T\nabla_{\sigma_1}\alpha_1^1 - (v_4 - v_3)\nabla_{\sigma_1}\kappa_1^1 - v_5^T\nabla_{\sigma_1}\alpha_2^1 = 0. \tag{5.21}$$

To see the implications of (5.21) first differentiate the budget constraint of generation one with respect to σ_1 to obtain

$$\pi_1^T\nabla_{\sigma_1}\alpha_1^1 + \rho_1\nabla_{\sigma_1}\kappa_1^1 = 0. \tag{5.22}$$

Substituting this into (5.21) and rearranging terms yields

$$\xi_2\frac{1}{\rho_1}\pi_1^T\alpha_1^1 = \left[v_2^T + (v_4 - v_3)\frac{\pi_1^T}{\rho_1}\right]\nabla_{\sigma_1}\alpha_1^1 + v_5^T\nabla_{\sigma_1}\alpha_2^1. \tag{5.23}$$

A necessary condition for all eight normalizations to be harmless yields in addition to the above that

$$v^T = \left[\mu_1(p_1^T, r_1), \mu_2(s_1, p_2^T, r_2), \mu_3(s_2, p_3^T)\right] \tag{5.24}$$

where μ_t is a function only of period t producer prices; as mentioned above, this follows from a similar argument in Section 5 of Blackorby and Brett [1999]. Thus, (5.23) can be rewritten as

$$\xi_2\frac{1}{\rho_1}\pi_1^T\alpha_1^1 = \left[\mu_1 p_1^T + (\mu_2 s_1 - \mu_1 r_1)\frac{\pi_1^T}{\rho_1}\right]\nabla_{\sigma_1}\alpha_1^1 + \mu_2 p_2^T\nabla_{\sigma_1}\alpha_2^1. \tag{5.25}$$

Carrying out the indicated differentiation and canceling ρ_1 yields

$$\xi_2\pi_1^T\alpha_1^1 = \left[\mu_1 p_1^T + (\mu_2 s_1 - \mu_1 r_1)\frac{\pi_1^T}{\rho_1}\right]\nabla_{\tilde{\pi}_1}\alpha_1^1\pi_1 + \mu_2 p_2^T\nabla_{\tilde{\pi}_1}\alpha_2^1\pi_1. \tag{5.26}$$

[26] This follows from the homogeneity of demand functions and the budget constraints.

Using (5.24) it follows from (5.19) that ξ_3 is homogeneous of degree minus one in (π_1, π_2, m) and from (5.13) it follows that ξ_2 has the same property. The left side of (5.26) is homogeneous of degree zero in (π_1, π_2, m). This is true of the right side if and only if $\mu_2 s_1 - \mu_1 r_1 = 0$; that is, we are in the region where the government has positive savings. This leaves us with

$$\xi_2 \pi_1^T \alpha_1^1 = \mu_1 p_1^T \nabla_{\tilde{\pi}_1} \alpha_1^1 \pi_1 + \mu_2 p_2^T \nabla_{\tilde{\pi}_1} \alpha_2^1 \pi_1. \tag{5.27}$$

If $\xi_2 > 0$, then dividing (5.25) by ξ_2 leaves the left side independent of producer prices while the right side depends upon them. Hence both sides must be zero which yields a contradiction because we have assumed positive savings by generations one and two and hence that $-\pi_1^T \alpha_1^1 = \rho_1 \kappa_1^1 > 0$. If $\xi_2 = 0$, then the right side of (5.25) must equal zero. Rearranging the right side of (5.25) so that we have but period one producer prices on one side of the equation and period two producer prices on the other side shows that

$$\nabla_{\tilde{\pi}_1} \alpha_1^1 \tilde{\pi} = 0 \quad \text{and} \quad \nabla_{\tilde{\pi}_1} \alpha_2^1 \tilde{\pi} = 0. \tag{5.28}$$

That is, the demand functions of generation one are homogeneous of degree zero in $\tilde{\pi}_1$ as well as being homogeneous of degree zero in all of its arguments. Thus, we have

$$\begin{aligned} \alpha_1^1(\tilde{\pi}_1, \pi_2, m) &= \alpha_1^1 \left(\frac{\tilde{\pi}_1}{m}, \pi_2, m \right) \\ &= \alpha_1^1 \left(\frac{\tilde{\pi}_1}{m^2}, \frac{\pi_2}{m}, 1 \right). \end{aligned} \tag{5.29}$$

From (5.29) it is clear that the demand functions are no longer homogeneous of degree zero in $(\tilde{\pi}_1, \pi_2, m)$ yielding a contradiction. We summarize this as

Theorem 1: *Given the regularity conditions, there are no conditions under which zero taxes on capital and savings hold identically as part of a Pareto-efficient solution in a finite-horizon OLG model with positive individual savings.*

This theorem depends explicitly on the OLG structure of the model. It is not the case that capital taxation is merely substituting for the lack of lump–sum transfers. For example, a captial tax can be used to adjust $\tilde{\pi}_1$. In order to simulate its effects we can normalize $\tilde{\pi}_1$, by dividing both sides of (2.4) by an appropriate constant. This operation has two effects: it changes the real income of generation one and inter–temporal prices it faces. Restrictions such as separability can be appealed to in order to rule out inter–temporal spillover effects on consumption for members of that generation. But the income effects remain. Lump–sum taxation is not available, so these income effects cannot be "purged". The argument expressed in (5.29) breaks down if $m = 0$. In this case, the income effects disappear. However, an analogous argument can be constructed by dividing through by the first element, say, of $\tilde{\pi}_1$. Even this argument breaks down due to the free normalization of a consumer price in period one. But there is no such free normalization of second period prices. The reason is quite simple: the budget constraint of generation one, which is still alive in period two, has already been normalized by the rescaling of first period prices. A second rescaling of its budget constraint is not possible. Any change in consumer prices in period 2 affects the real incomes of the two generations alive at that date. The government cannot adjust a single demogrant to simultaneously offset these two effects.

Theorem 1 does not claim that there are no Pareto efficient tax structures with zero captial taxation. The Walrasian equilibrium with neither taxes nor a demogrant is Pareto efficient. However, it is a *single point* on the Pareto frontier. Even points on the Pareto frontier near the laissez faire outcome require some form of capital market intervention.

6. Concluding Remarks

Capital taxation can have far–reaching effects on the economy. Like all forms of taxation, it affects the relative prices consumers face, and the real incomes at their disposal. By their very nature, the incidence of capital income taxes differs among agents. In the absence of optimal inter–generational transfer schemes, this differential incidence can be exploited to implement some parts of the Pareto frontier that would otherwise be unattainable. This paper has uncovered, perhaps surprisingly, that this intuition is more robust than it might first appear. No set of restrictions on preferences is sufficient to render the possible redistributive effects of capital taxation inoperable. Only a severe restriction on policy — that it not exist at all — suffices.

7. Appendix

This sections contains several technical arguments used in the text.

7.1. *Pareto-efficient Taxes*

Expanding (4.9) and (4.1) yields

$$\sum_t \xi_t - \mu_1 p_1^T \nabla_m \alpha_1 + (\mu_2 s_1 - \mu_1 r_1) \nabla_m \kappa_1^1 - \mu_2 p_2^T \nabla_m \alpha_2 + (\mu_3 s_2 - \mu_2 r_2) \nabla_m \kappa_2^2 - \mu_3 p_3^T \nabla_m \alpha_3^2 = 0, \tag{7.1}$$

$$-\xi_1 \alpha_1^{0T} - \xi_2 \frac{\sigma_1}{\rho_1} \alpha_1^{1T} - \mu_1 p_1^T \nabla_{\pi_1} \alpha_1 + (\mu_2 s_1 - \mu_1 r_1) \nabla_{\pi_1} \kappa_1^1 - \mu_2 p_2^T \nabla_{\pi_1} \alpha_2^1 = 0, \tag{7.2}$$

$$\xi_2 \frac{\sigma_1}{\rho_1^2} \pi_1^T \alpha_1^1 - \mu_1 p_1^T \nabla_{\rho_1} \alpha_1^1 + (\mu_2 s_1 - \mu_1 r_1) \nabla_{\rho_1} \kappa_1^1 - \mu_2 p_2^T \nabla_{\rho_1} \alpha_2^1 = 0, \tag{7.3}$$

$$-\xi_2 \frac{1}{\rho_1} \pi_1^T \alpha_1^1 - \mu_1 p_1^T \nabla_{\sigma_1} \alpha_1^1 + (\mu_2 s_1 - \mu_1 r_1) \nabla_{\sigma_1} \kappa_1^1 - \mu_2 p_2^T \nabla_{\sigma_1} \alpha_2^1 = 0, \tag{7.4}$$

$$\begin{aligned} -\xi_2 \alpha_2^{1T} - \xi_3 \frac{\sigma_2}{\rho_2} \alpha_2^{2T} - \mu_1 p_1^T \nabla_{\pi_2} \alpha_1^1 + (\mu_2 s_1 - \mu_1 r_1) \nabla_{\pi_2} \kappa_1^1 - \mu_2 p_2^T \nabla_{\pi_2} \alpha_2 \\ + (\mu_3 s_2 - \mu_2 r_2) \nabla_{\pi_2} \kappa_2^2 - \mu_3 p_3^T \nabla_{\pi_2} \alpha_3^2 = 0, \end{aligned} \tag{7.5}$$

$$\xi_3 \frac{\sigma_2}{\rho_2^2} \pi_2^T \alpha_2^2 - \mu_2 p_2^T \nabla_{\rho_2} \alpha_2^2 + (\mu_3 s_2 - \mu_2 r_2) \nabla_{\rho_2} \kappa_2^2 - \mu_3 p_3^T \nabla_{\rho_2} \alpha_2^2 = 0, \tag{7.6}$$

$$-\xi_3 \frac{1}{\rho_2} \pi_2^T \alpha_2^2 - \mu_2 p_2^T \nabla_{\sigma_2} \alpha_2^2 + (\mu_3 s_2 - \mu_2 r_2) \nabla_{\sigma_2} \kappa_2^2 - \mu_3 p_3^T \nabla_{\sigma_2} \alpha_2^2 = 0, \tag{7.7}$$

and

$$-\xi_3 \alpha_3^{2T} - \mu_2 p_2^T \nabla_{\pi_3} \alpha_2^2 + (\mu_3 s_2 - \mu_2 r_2) \nabla_{\pi_3} \kappa_2^2 - \mu_3 p_3^T \nabla_{\pi_3} \alpha_3^2 = 0. \tag{7.8}$$

Simple manipulation shows that (7.3) and (7.4) are not independent; similarly for (7.6) and (7.7). The set of Pareto optimal taxes and demogrant can be characterized by the equations in the text.

7.2. Normalizations

This model admits six independent price normalizations: one producer price per period can be fixed; one consumer price — but not one per period — may be fixed; and either the capital input taxes or the taxes on savings (but not both) may be set to zero.[27] We show that here only six are possible:

$$p_{11} = 1, \quad s_1 = 1, \quad s_2 = 1, \quad \tau^a_{11} = 0 \quad \tau^k_1 = 0, \quad \textit{and} \quad t^k_2 = 0. \tag{7.9}$$

Given our tax reform perspective, it is necessary to translate the normalizations and (possibly) binding restrictions into statements about the possible directions of policy reform. Clearly, the components of γ corresponding to a change in a normalized quantity must be zero. We can introduce these restrictions with the help of the following matrices:

$$\mathcal{I} = \begin{bmatrix} 1, 0^T_n & 0 & 0 & 0^T_n & 0 & 0 & 0^T_n \\ 0^T_n & 0 & 1 & 0^T_n & 0 & 0 & 0^T_n \\ 0^T_n & 0 & 0 & 0^T_n & 0 & 1 & 0^T_n \end{bmatrix}. \tag{7.10}$$

and

$$\tilde{\mathcal{I}} = \begin{bmatrix} 1, 0^T_{n-1} & 0 & 0 & 0^T_n & 0 & 0 & 0^T_n \\ 0^T_n & 0 & 1 & 0^T_n & 0 & 0 & 0^T_n \\ 0^T_n & 0 & 0 & 0^T_n & 0 & 1 & 0^T_n \end{bmatrix}. \tag{7.11}$$

The rows of the matrices correspond to the order in which the constraints are imposed by (7.9); for example, the first row of $\tilde{\mathcal{I}}$ imposes the constraint $d\tau^a_{11} = 0$.

[27] In Blackorby and Brett [1999], Theorem 1 it was shown that there are seven normalizations when there are generation specific transfer. In that case, the tax on the initial stock was equivalent to a change in the lump-sum tax-subsidy to generation zero.

These normalizations are imposed by

$$\begin{bmatrix}\mathcal{I}\\ \mathbf{0}\end{bmatrix}\gamma_p+\begin{bmatrix}\mathbf{0}\\ \tilde{\mathcal{I}}\end{bmatrix}\gamma_\tau+\begin{bmatrix}\mathbf{0}\\ \mathbf{0}\end{bmatrix}\gamma_m+\begin{bmatrix}\mathbf{0}\\ \mathbf{0}\end{bmatrix}\gamma_\kappa+\begin{bmatrix}\mathbf{0}\\ \mathbf{0}\end{bmatrix}\gamma_\eta=0. \tag{7.12}$$

There are strict Pareto-improving changes that are simultaneously equilibrium-preserving with three producer price normalizations and three capital inputs price normalizations and two consumer price normalizations if and only (3.15), (3.16), and (7.12) have a solution. If there is no such solution we are at a second-best optimum. Using Motzkin's Theorem the economy is at a second-best optimum if and only if

$$\begin{bmatrix}\xi^T & \theta\end{bmatrix}\begin{bmatrix}[P_\pi & P_\pi & P_m & \mathbf{0} & 0\\ 0_n^T & 0_n^T & 0 & 0_2^T & 1\end{bmatrix}+\begin{bmatrix}v^T & \eta^T\end{bmatrix}\begin{bmatrix}E_\pi+E_p & E_\pi & E_m & E_\kappa & 0\\ 0_{2\times n} & 0_{2\times n} & 0_2 & I_{2\times 2} & \kappa^g\end{bmatrix}$$

$$+\,(w^T,z^T)\begin{bmatrix}\mathcal{I} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0}\\ \mathbf{0} & \tilde{\mathcal{I}} & \mathbf{0} & \mathbf{0} & \mathbf{0}\end{bmatrix}=0 \tag{7.13}$$

where $0\neq[\xi^T,\theta]\geq 0^T$ and $[v^T,\eta^T]\geq 0^T$.

Expanding (7.13) yields

$$\xi^T P_\pi+v^T(E_\pi+E_p)+w^T\mathcal{I}=0, \tag{7.14}$$

$$\xi^T P_\pi+v^T E_\pi+z^T\tilde{\mathcal{I}}=0, \tag{7.15}$$

$$\xi^T P_m+v^T E_m=0, \tag{7.16}$$

$$v^T E_\kappa+\eta^T=0, \tag{7.17}$$

$$\theta+\eta^T\kappa^g=0. \tag{7.18}$$

Let $v^T=(v_1,v_2^T,v_3,v_4,v_5^T,v_6,v_7,v_8^T)$ where the second, fifth, and last element are n-tuples. Expanding yields (7.16) and then (7.15) yields

$$\sum_t\xi_t-v_2^T\nabla_m\alpha_1+(v_4-v_3)\nabla_m\kappa_1^1-v_5^T\nabla_m\alpha_2+(v_7-v_6)\nabla_m\kappa_2^2-v_8^T\nabla_m\alpha_3^2=0, \tag{7.19}$$

$$-\xi_1\alpha_1^{0T}-\xi_2\frac{\sigma_1}{\rho_1}\alpha_1^{1T}-v^T\nabla_{\pi_1}\alpha_1+(v_4-v_3)\nabla_{\pi_1}\kappa_1^1-v_5^T\nabla_{\pi_1}\alpha_2^1+z_1=0, \tag{7.20}$$

$$\xi_2 \frac{\sigma_1}{\rho_1^2} \pi_1^T \alpha_1^1 - v_2^T \nabla_{\rho_1} \alpha_1^1 + (v_4 - v_3) \nabla_{\rho_1} \kappa_1^1 - v_5^T \nabla_{\rho_1} \alpha_2^1 = 0, \tag{7.21}$$

$$-\xi_2 \frac{1}{\rho_1} \pi_1^T \alpha_1^1 - v_2^T \nabla_{\sigma_1} \alpha_1^1 + (v_4 - v_3) \nabla_{\sigma_1} \kappa_1^1 - v_5^T \nabla_{\sigma_1} \alpha_2^1 + z_2 = 0, \tag{7.22}$$

$$-\xi_2 \alpha_2^{1T} - \xi_3 \frac{\sigma_2}{\rho_2} \alpha_2^{2T} - v_2^T \nabla_{\pi_2} \alpha_1^1 + (v_4 - v_3) \nabla_{\pi_2} \kappa_1^1 - v_5^T \nabla_{\pi_2} \alpha_2 + (v_7 - v_6) \nabla_{\pi_2} \kappa_2^2 - v_8^T \nabla_{\pi_2} \alpha_3^2 = 0, \tag{7.23}$$

$$\xi_3 \frac{\sigma_2}{\rho_2^2} \pi_2^T \alpha_2^2 - v_5^T \nabla_{\rho_2} \alpha_2^2 + (v_7 - v_6) \nabla_{\rho_2} \kappa_2^2 - v_8^T \nabla_{\rho_2} \alpha_2^2 = 0, \tag{7.24}$$

$$-\xi_3 \frac{1}{\rho_2} \pi_2^T \alpha_2^2 - v_5^T \nabla_{\sigma_2} \alpha_2^2 + (v_7 - v_6) \nabla_{\sigma_2} \kappa_2^2 - v_8^T \nabla_{\sigma_2} \alpha_2^2 + z_3 = 0, \tag{7.25}$$

and

$$-\xi_3 \alpha_3^{2T} - v_5^T \nabla_{\pi_3} \alpha_2^2 + (v_7 - v_6) \nabla_{\pi_3} \kappa_2^2 - v_8^T \nabla_{\pi_3} \alpha_3^2 = 0. \tag{7.26}$$

Multiplying (7.21) by ρ_1, (7.22) by σ_1, adding and using the homogeneity of κ_1^1 yields

$$\sigma_1 z_2 = 0 \tag{7.27}$$

so that z_2 is identically zero and the constraint that $t_1^k = 0$ is not binding; hence a free normalization. Similarly multiply (7.24) by ρ_2, (7.23) by σ_2, adding and using the homogeneity of κ_2^2 yields

$$\sigma_2 z_3 = 0 \tag{7.28}$$

so that z_3 is identically zero and the constraint that $t_2^k = 0$ is not binding. Next, multiply (7.19) by m, (7.20) by π_1, (7.23) by π_2, (7.26) by π_3, add, use the homogeneity of the demand equations and the budget constraints to obtain

$$z_1 \pi_{11} = 0. \tag{7.29}$$

Thus the tax on one consumer price can be set equal to zero. Next, following the argument in Subsection 4.1 of Blackorby and Brett [1999] shows that one producer price can be normalized in each period.

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