THE CONSUMER PRICE INDEX AND INDEX NUMBER THEORY: A SURVEY

by

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Abstract:

The paper addresses the problem of choosing the "best" functional form for the Consumer Price Index from the viewpoint of five different approaches to index number theory. The five approaches are: (1) fixed basket approaches; (2) the Divisia approach; (3) the axiomatic or test approach; (4) the stochastic approach and (5) the economic approach. It turns out that the Divisia approach does not lead to a definite choice of index number formula but the other 4 approaches lead to 3 or 4 formulae as "best" choices. These index number formula turn all turn out to be superlative index number formulae that approximate each other quite closely. The paper also considers the problem of obtaining "practical" approximations to these superlative indexes that can be implemented in a timely fashion.

Journal of Economic Literature Classification Numbers: C43; D11; E31; O47.

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A. Introduction

"The answer to the question what is the Mean of a given set of magnitudes cannot in general be found, unless there is given also the object for the sake of which a mean value is required. There are as many kinds of average as there are purposes; and we may almost say in the matter of prices as many purposes as writers. Hence much vain controversy between persons who are literally at cross purposes." F.Y. Edgeworth (1888; 347).

1. The number of physically distinct goods and unique types of services that consumers can purchase is in the millions. On the business or production side of the economy, there are even more commodities that are actively traded. This is because firms not only produce commodities for final consumption, they also produce exports and intermediate commodities that are demanded by other producers. Firms collectively also use millions of imported goods and services, thousands of different types of labor services and hundreds of thousands of specific types of capital. If we further distinguish physical commodities by their geographic location or by the season or time of day that they are produced or consumed, then there are *billions* of commodities that are traded within each year of any advanced economy. For many purposes, it is necessary to *summarize* this

vast amount of price and quantity information into a much smaller set of numbers. The question that this chapter addresses is: *how exactly should the microeconomic information involving possibly millions of prices and quantities be aggregated into a smaller number of price and quantity variables?* This is the basic *index number problem.*

2. It is possible to pose the index number problem in the context of microeconomic theory; i.e., given that we wish to implement some economic model based on producer or consumer theory, what is the "best" method for constructing a set of aggregates for the model? However, when constructing aggregate prices or quantities, other points of view (that do not rely on economics) are possible. We will also consider some of these alternative points of view in this chapter. Thus in sections B to G below, we consider some of the early noneconomic approaches to index number theory. Economic approaches are considered in sections H and I below.

3. The index number problem can be framed as the problem of decomposing the value of a well defined set of transactions in a period of time into an aggregate price term times an aggregate quantity term. It turns out that this approach to the index number problem does not lead to any useful solutions so in section B below, we consider the problem of decomposing a *value ratio* pertaining to two periods of time into a component that measures the overall *change in prices* between the two periods (this is the *price index*) times a term that that measures the overall *change in quantities* between the two periods (this is the *quantity index*). The simplest price index is *a fixed basket type index*; i.e., we choose fixed amounts of the n quantities in the value aggregate and price this fixed basket of quantities at the prices of period 0 and at the prices of period 1 and our fixed basket price index is simply the ratio of these two values where prices vary but quantities are held fixed. Two natural choices for the fixed basket are the quantities transacted in the base period, period 0, or the quantities transacted in the current period, period 1. These two choices lead to the Laspeyres (1871) and Paasche (1874) price indices respectively. Appendix 3.1 provides a decomposition of the difference between these two indices into "explanatory" factors.

4. Unfortunately, the Paasche and Laspeyres measures of aggregate price change can differ, sometimes, substantially. Thus in section C, we consider taking an average of these two indices to come up with a single measure of price change. In section C.1, we argue that the "best" average to take is the geometric mean, which is Irving Fisher's (1922) ideal price index. In section C.2, instead of averaging the Paasche and Laspeyres measures of price change, we consider taking an average of the two baskets. This fixed basket approach to index number theory leads to a price index advocated by Walsh (1901) (1921).

5. In section D, we consider another approach to the determination of the *functional form* or the *formula* for the price index. This approach is due to the French economist, Divisia (1926), and his approach is based on the assumption that price and quantity data are available as continuous functions of time. The theory of differentiation is used in order to decompose the rate of change of a continuous time value aggregate into two components that reflect aggregate price and quantity change. At first sight, it appears that

this approach does not have any connection with the other approaches to index number theory but Appendix 3.2 shows that the Divisia approach does have a connection with the economic approach. Although the approach of Divisia offers some insights[3], it does not offer much guidance to statistical agencies in terms of leading to a *definite* choice of index number formula.

6. In section E, we consider the advantages and disadvantages of using a *fixed base* period in the bilateral index number comparison versus always comparing the current period with the previous period, which is called the *chain system*. In the chain system, a *link* is an index number comparison of one period with the previous period. These links are multiplied together in order to make comparisons over many periods.

7. Having considered various fixed basket approaches to index number theory as well as the approach of Divisia, in section F, we consider the third major noneconomic approach to index number theory, which is the *test* or *axiomatic approach*. In this approach, we attempt to determine the functional form for the price and quantity indices by asking that these aggregation functions have various intuitively plausible *properties*. It appears that the Fisher ideal price index does "best" from this viewpoint.

8. In section G, we consider the fourth major noneconomic approach to index number theory, the *stochastic approach*. In section G.1, we consider the very early stochastic approach due to Jevons (1865) (1884) and Edgeworth (1888) (1923). These early approaches assumed that an adequate price index could be obtained simply by taking the average of a large number of price relatives of the form p_i^1/p_i^0, where p_i^t is the price of commodity i in period t. This early stochastic approach is not considered suitable for statistical agencies because it does not take into account the economic importance of the individual price quotations, p_i^t. However, Theil (1967) worked out a weighted stochastic approach to index number theory that is very suitable for statistical agencies. His approach leads to a functional form for the price index that was advocated by the Finnish economist, Törnqvist (1936).

9. In section H, we turn to our fifth and last approach to index number theory, the *economic approach*. In section H, we consider the case where there is only one (representative) household while in section I, we consider the many households case. In the economic approach, it is assumed that the observed price and quantity data are generated as solutions to various economic optimization problems. Many price statisticians find the assumptions made in the economic approach to be somewhat implausible. Perhaps the best way to regard the assumptions made in the economic approach is that they simply formalize the fact that consumers tend to purchase more of a commodity if its price falls relative to other prices. In section H, it will be shown that the Fisher, Walsh and Törnqvist price indices (which emerged as being best in the various noneconomic approaches) are also among the "best" in the economic approach to index number theory. Moreover, it will be shown in Appendix 3.3 that these three indices

[3] In particular, it can be used to justify the chain system of index numbers, which will be discussed in section E.

approximate each other very closely using "normal" time series data. Thus as far as index number *theory* is concerned, it appears at this stage that "all roads lead to Rome".

10. Section I concludes this chapter with a discussion of the economic approach to index number theory when there are many households. In particular, the theory of democratic and plutocratic price indices initiated by Prais (1959) is discussed.

B. The decomposition of value aggregates into price and quantity components

B.1 The decomposition of value aggregates and the product test

11. A *price index* is a measure or function which summarizes the *change* in the prices of many commodities from one situation 0 (a time period or place) to another situation 1. More specifically, for most practical purposes, a price index can be regarded as a weighted mean of the change in the relative prices of the commodities under consideration in the two situations. To determine a price index, we need to know:

- which commodities or items to include in the index;
- how to determine the item prices;
- which transactions that involve these items to include in the index;
- how to determine the weights and from which sources should these weights be drawn;
- what formula or type of mean should be used to average the selected item relative prices.

All of the above price index definition questions except the last can be answered by appealing to the definition of the *value aggregate* to which the price index refers. A *value aggregate* V for a given collection of items and transactions is computed as:

(3.1) $V = \sum_{i=1}^{n} p_i q_i$

where p_i represents the price of the ith item in national currency units, q_i represents the corresponding quantity transacted in the time period under consideration and the subscript i identifies the ith elementary item in the group of n items that make up the chosen value aggregate V. Included in this definition of a value aggregate is the specification of the group of included commodities (which items to include) and of the economic agents engaging in transactions involving those commodities (which transactions to include), as well as the valuation and time of recording principles motivating the behavior of the economic agents undertaking the transactions (determination of prices). The included elementary items, their valuation (the p_i), the eligibility of the transactions and the item weights (the q_i) are all within the domain of definition of the value aggregate. The precise determination of the p_i and q_i will be discussed in more detail in subsequent chapters.[4]

[4] Ralph Turvey has reminded us that some values may be difficult to decompose into unambiguous price and quantity components. Some examples of difficult to decompose values are bank charges, gambling expenditures and life insurance payments.

12. The value aggregate V defined by (3.1) above referred to a certain set of transactions pertaining to a single (unspecified) time period. We now consider the same value aggregate for two places or time periods, periods 0 and 1. For the sake of definiteness, we call period 0 *the base period* and period 1 *the current period* and we assume that we have collected observations on the base period price and quantity vectors, $p^0 \equiv [p_1^0,\ldots,p_n^0]$ and $q^0 \equiv [q_1^0,\ldots,q_n^0]$ respectively, as well as on the current period price and quantity vectors, $p^1 \equiv [p_1^1,\ldots,p_n^1]$ and $q^1 \equiv [q_1^1,\ldots,q_n^1]$ respectively.[5] The value aggregates in the two periods are defined in the obvious way as:

$$(3.2)\quad V^0 \equiv \sum_{i=1}^n p_i^0\, q_i^0\ ;\ V^1 \equiv \sum_{i=1}^n p_i^1\, q_i^1\ .$$

In the previous paragraph, we defined a price index as a function or measure which summarizes the *change* in the prices of the n commodities in the value aggregate from situation 0 to situation 1. In this paragraph, we will be more general and define a *price index* $P(p^0,p^1,q^0,q^1)$ along with the corresponding *quantity index* (or *volume index*) $Q(p^0,p^1,q^0,q^1)$ to be two functions of the 4n variables p^0,p^1,q^0,q^1 (these variables describe the prices and quantities pertaining to the value aggregate for periods 0 and 1) where these two functions satisfy the following equation:[6]

$$(3.3)\quad V^1/V^0 = P(p^0,p^1,q^0,q^1)\, Q(p^0,p^1,q^0,q^1).$$

If there is only one item in the value aggregate, then the price index P should collapse down to the single price ratio, p_1^1/p_1^0 and the quantity index Q should collapse down to the single quantity ratio, q_1^1/q_1^0. In the case of many items, the price index P is to be interpreted as some sort of weighted average of the individual price ratios, $p_1^1/p_1^0,\ldots, p_n^1/p_n^0$.

13. Thus our first approach to index number theory can be regarded as the problem of *decomposing* the change in a value aggregate, V^1/V^0, into the product of a part that is due to *price change*, $P(p^0,p^1,q^0,q^1)$, and a part that is due to *quantity change*, $Q(p^0,p^1,q^0,q^1)$. This approach to the determination of the price index is the approach that is taken in the national accounts, where a price index is used to *deflate* a value ratio in order to obtain an estimate of quantity change. Thus in this approach to index number theory, the primary use for the price index is as a *deflator*. Note that once the functional form for the price index $P(p^0,p^1,q^0,q^1)$ is known, then the corresponding quantity or volume index $Q(p^0,p^1,q^0,q^1)$ is completely determined by P; i.e., rearranging (3.3), we have:

$$(3.4)\quad Q(p^0,p^1,q^0,q^1) = [V^1/V^0]/P(p^0,p^1,q^0,q^1).$$

Conversely, if the functional form for the quantity index $Q(p^0,p^1,q^0,q^1)$ is known, then the corresponding price index $P(p^0,p^1,q^0,q^1)$ is completely determined by Q. Thus using this deflation approach to index number theory, we do not require separate theories for the

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[5] Note that we are assuming that there are no new or disappearing commodities in the value aggregates. Approaches to the "new goods problem" will be discussed in Chapter 4.

[6] The first person to suggest that the price and quantity indices should be jointly determined in order to satisfy equation 3.3 was Fisher (1911; 418). Frisch (1930; 399) called 3.3 the *product test*.

determination of the price and quantity indices: if we have determined either P or Q, then the other function is implicitly determined by the product test (3.3).

14. In the next subsection, we will consider two concrete choices for the price index $P(p^0,p^1,q^0,q^1)$ and calculate the corresponding quantity indices $Q(p^0,p^1,q^0,q^1)$ that result from using equation (3.4). These are the two choices used most frequently by national income accountants.

B.2 The Laspeyres and Paasche indices

15. One of the simplest approaches to the determination of the price index formula was described in great detail by Lowe (1823). His approach to measuring the price change between periods 0 and 1 was to specify an approximate *representative commodity basket*[7], which is a quantity vector $q \equiv [q_1,\ldots,q_n]$, and then calculate the level of prices in period 1 relative to period 0 as the ratio of the period 1 cost of the basket, $\sum_{i=1}^n p_i^1 q_i$, to the period 0 cost of the basket, $\sum_{i=1}^n p_i^0 q_i$. This *fixed basket approach* to the determination of the price index leaves open the question as to how exactly is the fixed basket vector q to be chosen?

16. As time passed, economists and price statisticians demanded a bit more precision with respect to the specification of the basket vector q. There are two natural choices for the reference basket: the base period 0 commodity vector q^0 or the current period 1 commodity vector q^1. These two choices lead to the *Laspeyres* (1871) price index[8] P_L defined by (3.5) and the *Paasche* (1874) price index[9] P_P defined by (3.6):[10]

$$(3.5)\quad P_L(p^0,p^1,q^0,q^1) \equiv \sum_{i=1}^n p_i^1 q_i^0 / \sum_{i=1}^n p_i^0 q_i^0 ;$$

$$(3.6)\quad P_P(p^0,p^1,q^0,q^1) \equiv \sum_{i=1}^n p_i^1 q_i^1 / \sum_{i=1}^n p_i^0 q_i^1 .$$

17. The above formulae can be rewritten in an alternative manner that is very useful for statistical agencies. Define the period t expenditure share on commodity i as follows:

$$(3.7)\quad s_i^t \equiv p_i^t q_i^t / \sum_{j=1}^n p_j^t q_j^t \text{ for } i = 1,\ldots,n \text{ and } t = 0,1.$$

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[7] Lowe (1823; Appendix page 95) suggested that the commodity basket vector q should be updated every five years.

[8] This index was actually introduced and justified by Drobisch (1871a; 147) slightly earlier than Laspeyres. Laspeyres (1871; 305) in fact explicitly acknowledged that Drobisch showed him the way forward. However, the contributions of Drobisch have been forgotten for the most part by later writers because Drobisch aggressively pushed for the ratio of two unit values as being the “best” index number formula. While this formula has some excellent properties if all of the n commodities being compared have the same unit of measurement, the formula is useless when say, both goods and services are in the index basket.

[9] Again Drobisch (1871b; 424) appears to have been the first to define explicitly and justify this formula. However, he rejected this formula in favor of his preferred formula, the ratio of unit values, and so again he did not get any credit for his early suggestion of the Paasche formula.

[10] Note that $P_L(p^0,p^1,q^0,q^1)$ does not actually depend on q^1 and $P_P(p^0,p^1,q^0,q^1)$ does not actually depend on q^0. However, it does no harm to include these vectors and the notation indicates that we are in the realm of bilateral index number theory; i.e., we are comparing the prices and quantities for a value aggregate pertaining to two periods.

Then the Laspeyres index (3.5) can be rewritten as follows:

$$(3.8)\quad P_L(p^0,p^1,q^0,q^1) = \sum_{i=1}^n p_i^1 q_i^0 / \sum_{j=1}^n p_j^0 q_j^0$$
$$= \sum_{i=1}^n (p_i^1/p_i^0)\, p_i^0 q_i^0 / \sum_{j=1}^n p_j^0 q_j^0$$
$$= \sum_{i=1}^n (p_i^1/p_i^0)\, s_i^0 \qquad \text{using definitions (3.7).}$$

Thus the Laspeyres price index P_L can be written as a base period expenditure share weighted arithmetic average of the n price ratios, p_i^1/p_i^0 . The Laspeyres formula (until the very recent past) has been widely used as the intellectual base for Consumer Price Indices around the world. To implement it, a statistical agency need only collect information on expenditure shares s_n^0 for the index domain of definition for the base period 0 and then collect information on item *prices* alone on an ongoing basis. *Thus the Laspeyres CPI can be produced on a timely basis without having to know current period quantity information.*

18. The Paasche index can also be written in expenditure share and price ratio form as follows:

$$(3.9)\quad P_P(p^0,p^1,q^0,q^1) = 1/ [\sum_{i=1}^n p_i^0 q_i^1 / \sum_{j=1}^n p_j^1 q_j^1]$$
$$= 1/ [\sum_{i=1}^n (p_i^0/p_i^1)\, p_i^1 q_i^1 / \sum_{j=1}^n p_j^1 q_j^1]$$
$$= 1/ [\sum_{i=1}^n (p_i^1/p_i^0)^{-1} s_i^1] \qquad \text{using definitions (3.7)}$$
$$= [\sum_{i=1}^n (p_i^1/p_i^0)^{-1} s_i^1]^{-1}.$$

Thus the Paasche price index P_P can be written as a period 1 (or current period) expenditure share weighted *harmonic* average of the n item price ratios p_i^1/p_i^0.[11] The lack of information on current period quantities prevents statistical agencies from producing Paasche indices on a timely basis.

19. The quantity index that corresponds to the Laspeyres price index using the product test (3.3) is the Paasche quantity index; i.e., if we replace P in (3.4) by P_L defined by (3.5), we obtain the following quantity index:

$$(3.10)\quad Q_P(p^0,p^1,q^0,q^1) \equiv \sum_{i=1}^n p_i^1 q_i^1 / \sum_{i=1}^n p_i^1 q_i^0 .$$

Note that Q_P is the value of the period 1 quantity vector valued at the period 1 prices, $\sum_{i=1}^n p_i^1 q_i^1$, divided by the (hypothetical) value of the period 0 quantity vector valued at the period 1 prices, $\sum_{i=1}^n p_i^1 q_i^0$. Thus the period 0 and 1 quantity vectors are valued at the same set of prices, the current period prices, p^1.

20. The quantity index that corresponds to the Paasche price index using the product test (3.3) is the Laspeyres quantity index; i.e., if we replace P in (3.4) by P_P defined by (3.6), we obtain the following quantity index:

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[11] Note that the derivation in (3.9) shows how harmonic averages arise in index number theory in a very natural way.

(3.11) $Q_L(p^0,p^1,q^0,q^1) \equiv \sum_{i=1}^n p_i^0q_i^1 / \sum_{i=1}^n p_i^0q_i^0$.

Note that Q_L is the (hypothetical) value of the period 1 quantity vector valued at the period 0 prices, $\sum_{i=1}^n p_i^0q_i^1$, divided by the value of the period 0 quantity vector valued at the period 0 prices, $\sum_{i=1}^n p_i^0q_i^0$. Thus the period 0 and 1 quantity vectors are valued at the same set of prices, the base period prices, p^0.

21. The problem with the Laspeyres and Paasche index number formulae is that they are equally plausible but in general, they will give *different* answers. For most purposes, it is not satisfactory for the statistical agency to provide *two* answers to the question: what is the “best” overall summary measure of price change for the value aggregate over the two periods in question? Thus in the following section, we consider how “best” averages of these two estimates of price change can be constructed. What is the “normal” relationship between the Paasche and Laspeyres indices? Under “normal” economic conditions when the price ratios pertaining to the two situations under consideration are negatively correlated with the corresponding quantity ratios, it can be shown that the Laspeyres price index will be larger than the corresponding Paasche index.[12] In Appendix 3.1 below, we provide a precise statement of this result.[13] This divergence between P_L and P_P suggests that if we require a *single estimate* for the price change between the two periods, then we should take some sort of evenly weighted average of the two indices as our final estimate of price change between periods 0 and 1. As mentioned above, we will pursue this strategy in the following section. However, it should be kept in mind that usually, statistical agencies will not have information on current expenditure weights and hence averages of Paasche and Laspeyres indices can only be produced on a delayed basis (perhaps using national accounts information) or not at all.

C. Symmetric averages of fixed basket price indices

C.1 The Fisher index as an average of the Paasche and Laspeyres indices

22. As was mentioned in the previous paragraph, since the Paasche and Laspeyres price indices are equally plausible but often give different estimates of the amount of aggregate

[12] Peter Hill (1993; 383) summarizes this inequality as follows: “It can be shown that relationship (13) [i.e., that P_L is greater than P_P] holds whenever the price and quantity relatives (weighted by values) are negatively correlated. Such negative correlation is to be expected for price takers who react to changes in relative prices by substituting goods and services that have become relatively less expensive for those that have become relatively more expensive. In the vast majority of situations covered by index numbers, the price and quantity relatives turn out to be negatively correlated so that Laspeyres indices tend systematically to record greater increases than Paasche with the gap between them tending to widen with time.”

[13] There is another way to see why P_P will often be less than P_L. If the period 0 expenditure shares s_i^0 are exactly equal to the corresponding period 1 expenditure shares s_i^1, then by Schlömilch's Inequality (see Hardy, Littlewood and Polyá (1934)), it can be shown that a weighted harmonic mean of n numbers is equal to or less than the corresponding arithmetic mean of the n numbers and the inequality is strict if the n numbers are not all equal. If expenditure shares are approximately constant across periods, then it follows that P_P will usually be less than P_L under these conditions.

price change between periods 0 and 1, it is useful to consider taking an evenly weighted average of these fixed basket price indices as a single estimator of price change between the two periods. Examples of such *symmetric averages*[14] are the arithmetic mean, which leads to the Drobisch (1871b; 425) Sidgwick (1883; 68) Bowley (1901; 227)[15] index, $P_D \equiv (1/2)P_L + (1/2)P_P$, and the geometric mean, which leads to the Fisher[16] (1922) ideal index, P_F defined as

$$(3.12)\quad P_F(p^0,p^1,q^0,q^1) \equiv [P_L(p^0,p^1,q^0,q^1)\, P_P(p^0,p^1,q^0,q^1)]^{1/2} .$$

At this point, the fixed basket approach to index number theory is transformed into the *test approach* to index number theory; i.e., in order to determine which of these fixed basket indices or which averages of them might be "best", we need *criteria* or *tests* or *properties* that we would like our indices to satisfy. We will pursue this topic in more detail in section F below but we provide an introduction to the test approach in the present section because we use a test to determine which average of the Paasche and Laspeyres indices might be "best".

23. What is the "best" symmetric average of P_L and P_P to use as a point estimate for the theoretical cost of living index? It is very desirable for a price index formula that depends on the price and quantity vectors pertaining to the two periods under consideration to satisfy the *time reversal test*[17]. We say that the index number formula $P(p^0,p^1,q^0,q^1)$ satisfies this test if

$$(3.13)\quad P(p^1,p^0,q^1,q^0) = 1/ P(p^0,p^1,q^0,q^1) \;;$$

i.e., if we interchange the period 0 and period 1 price and quantity data and evaluate the index, then this new index $P(p^1,p^0,q^1,q^0)$ is equal to the reciprocal of the original index $P(p^0,p^1,q^0,q^1)$. This is a property that is satisfied by a single price ratio and it seems desirable that our measure of aggregate price change should also satisfy this property so that it does not matter which period is chosen as the base period. Put another way, the index number comparison between any two points of time should not depend on the choice of which period we regard as the base period: if we choose the other period as the base period, then our new index number should simply equal the reciprocal of the original index. It should be noted that the Laspeyres and Paasche price indices *do not* satisfy this time reversal property.

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[14] For a discussion of the properties of symmetric averages, see Diewert (1993c). Formally, an average m(a,b) of two numbers a and b is symmetric if m(a,b) = m(b,a). In other words, the numbers a and b are treated in the same manner in the average. An example of a nonsymmetric average of a and b is (1/4)a + (3/4)b.

[15] See Diewert (1993a; 36) for additional references to the early history of index number theory.

[16] Bowley (1899; 641) appears to have been the first to suggest the use of this index.

[17] See Diewert (1992; 218) for early references to this test, which is discussed in more detail in section F.3 below. If we want our price index to have the same property as a single price ratio, then it is important to satisfy the time reversal test. However, other points of view are possible. For example, we may want to use our price index for compensation purposes in which case, satisfaction of the time reversal test is not so important.

24. Having defined what it means for a price index P to satisfy the time reversal test, then it is possible to establish the following result:[18] the Fisher ideal price index defined by (3.12) above is the *only* index that is a homogeneous[19] symmetric average of the Laspeyres and Paasche price indices, P_L and P_P, and satisfies the time reversal test (3.13) above. Thus the Fisher ideal price index emerges as perhaps the "best" evenly weighted average of the Paasche and Laspeyres price indices.

25. It is interesting to note that this *symmetric basket approach* to index number theory dates back to one of the early pioneers of index number theory, Bowley, as the following quotations indicate:

"If [the Paasche index] and [the Laspeyres index] lie close together there is no further difficulty; if they differ by much they may be regarded as inferior and superior limits of the index number, which may be estimated as their arithmetic mean … as a first approximation." A. L. Bowley (1901; 227).

"When estimating the factor necessary for the correction of a change found in money wages to obtain the change in real wages, statisticians have not been content to follow Method II only [to calculate a Laspeyres price index], but have worked the problem backwards [to calculate a Paasche price index] as well as forwards. … They have then taken the arithmetic, geometric or harmonic mean of the two numbers so found." A. L. Bowley (1919; 348).[20]

26. The quantity index that corresponds to the Fisher price index using the product test (3.3) is the Fisher quantity index; i.e., if we replace P in (3.4) by P_F defined by (3.12), we obtain the following quantity index:

$$(3.14)\quad Q_F(p^0,p^1,q^0,q^1) \equiv [Q_L(p^0,p^1,q^0,q^1)Q_P(p^0,p^1,q^0,q^1)]^{1/2} .$$

Thus the Fisher quantity index is equal to the square root of the product of the Laspeyres and Paasche quantity indices. It should also be noted that $Q_F(p^0,p^1,q^0,q^1) = P_F(q^0,q^1,p^0,p^1)$; i.e., if we interchange the role of prices and quantities in the Fisher price index formula, then we obtain the Fisher quantity index.[21]

27. Rather than take a symmetric average of the two basic fixed basket price indices pertaining to two situations, P_L and P_P, it is also possible to return to Lowe's basic formulation and choose the basket vector q to be a symmetric average of the base and current period basket vectors, q^0 and q^1. We pursue this approach to index number theory in the following subsection.

C.2 The Walsh index and the theory of the "pure" price index

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[18] See Diewert (1997; 138).

[19] An average or mean of two numbers a and b, m(a,b), is *homogeneous* if when we multiply both numbers a and b by a positive number λ, then the mean is also multiplied by λ; i.e., m satisfies the following property: $m(\lambda a,\lambda b) = \lambda m(a,b)$.

[20] Fisher (1911; 417-418) (1922) also considered the arithmetic, geometric and harmonic averages of the Paasche and Laspeyres indices.

[21] Fisher (1922; 72) said that P and Q satisfied the *factor reversal test* if $Q(p^0,p^1,q^0,q^1) = P(q^0,q^1,p^0,p^1)$ and P and Q satisfied the product test (3.3) as well.

28. Price statisticians tend to be very comfortable with a concept of the price index that is based on pricing out a constant "representative" basket of commodities, $q \equiv (q_1,q_2,\ldots,q_n)$, at the prices of period 0 and 1, $p^0 \equiv (p_1^0,p_2^0,\ldots,p_n^0)$ and $p^1 \equiv (p_1^1,p_2^1,\ldots,p_n^1)$ respectively. Price statisticians refer to this type of index as a *pure price index*[22] and it corresponds to Knibbs' (1924; 43) *unequivocal price index*.[23] Thus the general functional form for the *pure price index* is

$$(3.15)\quad P_K(p^0,p^1,q) \equiv \sum_{i=1}^n p_i^1 q_i / \sum_{i=1}^n p_i^0 q_i = \sum_{i=1}^n s_i(p_i^1/p_i^0)$$

where the (hypothetical) expenditure shares s_i corresponding to the quantity weights vector q are defined by:

$$(3.16)\quad s_i \equiv p_i^0 q_i / \sum_{j=1}^n p_j^0 q_j \qquad \text{for } i = 1,2,\ldots,n.$$

29. The main reason why price statisticians might prefer the family of pure or unequivocal price indices defined by (3.15) is *that the fixed basket concept is easy to explain to the public*. Note that the Laspeyres and Paasche indices are special cases of the pure price concept if we choose $q = q^0$ (which leads to the Laspeyres index) or if we choose $q = q^1$ (which leads to the Paasche index).[24] The practical problem of picking q remains to be resolved and that is the problem we will address in this section.

30. It should be noted that Walsh (1901) (1921) also saw the price index number problem in the above framework:

"Commodities are to be weighted according to their importance, or their full values. But the problem of axiometry always involves at least two periods. There is a first period, and there is a second period which is compared with it. Price-variations have taken place between the two, and these are to be averaged to get the amount of their variation as a whole. But the weights of the commodities at the second period are apt to be different from their weights at the first period. Which weights, then, are the right ones—those of the first period? Or those of the second? Or should there be a combination of the two sets? There is no reason for preferring either the first or the second. Then the combination of both would seem to be the proper answer. And this combination itself involves an averaging of the weights of the two periods." Correa Moylan Walsh (1921; 90).

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[22] See section 7 in Diewert (2001). This concept for a price index dates back to Lowe (1823) at least.

[14] "Suppose however that, for each commodity, Q′ = Q, then the fraction, ∑(P′Q) / ∑(PQ), viz., the ratio of aggregate value for the second unit-period to the aggregate value for the first unit-period is no longer merely a ratio of totals, it also shows unequivocally the effect of the change in price. Thus it is an unequivocal price index for the quantitatively unchanged complex of commodities, A, B, C, etc.

It is obvious that if the quantities were different on the two occasions, and if at the same time the prices had been unchanged, the preceding formula would become ∑(PQ′) / ∑(PQ). It would still be the ratio of the aggregate value for the second unit-period to the aggregate value for the first unit period. But it would be also more than this. It would show in a generalized way the ratio of the quantities on the two occasions. Thus it is an unequivocal quantity index for the complex of commodities, unchanged as to price and differing only as to quantity.

Let it be noted that the mere algebraic form of these expressions shows at once the logic of the problem of finding these two indices is identical." Sir George H. Knibbs (1924; 43-44).

[24] Note that the ith share defined by (3.16) in this case is the "mixed" share $s_i \equiv p_i^0 q_i^1 / \sum_{j=1}^n p_j^0 q_j^1$ which uses the prices of period 0 and the quantities of period 1.

We will follow Walsh's suggestion and restrict the ith quantity weight, q_i, to be an average or *mean* of the base period quantity q_i^0 and the current period quantity for commodity i q_i^1, say $m(q_i^0,q_i^1)$, for i = 1,2,…,n.[25] Under this assumption, the pure price index (3.15) becomes:

(3.17) $P_K(p^0,p^1,q^0,q^1) \equiv \sum_{i=1}^n p_i^1 m(q_i^0,q_i^1) / \sum_{j=1}^n p_j^0 m(q_j^0,q_j^1)$.

31. In order to determine the functional form for the mean function m, it is necessary to impose some *tests* or *axioms* on the pure price index defined by (3.17). As in section C.1, we ask that P_K satisfy the *time reversal test*, (3.13) above. Under this hypothesis, it is immediately obvious that the mean function m must be a *symmetric mean*[26]; i.e., m must satisfy the following property: $m(a,b) = m(b,a)$ for all $a > 0$ and $b > 0$. This assumption still does not pin down the functional form for the pure price index defined by (3.17) above. For example, the function m(a,b) could be the *arithmetic mean*, (1/2)a + (1/2)b, in which case (3.17) reduces to the *Marshall* (1887) *Edgeworth* (1925) *price index* P_{ME}, which was the pure price index preferred by Knibbs (1924; 56):

(3.18) $P_{ME}(p^0,p^1,q^0,q^1) \equiv \sum_{i=1}^n p_i^1(1/2)(q_i^0 + q_i^1) / \sum_{j=1}^n p_j^0(1/2)(q_j^0 + q_j^1)$.

32. On the other hand, the function m(a,b) could be the *geometric mean*, $(ab)^{1/2}$, in which case (3.17) reduces to the *Walsh* (1901; 398) (1921; 97) *price index*, P_W[27]:

(3.19) $P_W(p^0,p^1,q^0,q^1) \equiv \sum_{i=1}^n p_i^1(q_i^0q_i^1)^{1/2} / \sum_{j=1}^n p_j^0(q_j^0q_j^1)^{1/2}$.

33. There are many other possibilities for the mean function m, including the mean of order r, $[(1/2)a^r + (1/2)b^r]^{1/r}$ for $r \neq 0$. Obviously, in order to completely determine the functional form for the pure price index P_K, we need to impose at least one additional test or axiom on $P_K(p^0,p^1,q^0,q^1)$.

34. There is a potential problem with the use of the Edgeworth Marshall price index (3.18) that has been noticed in the context of using the formula to make international comparisons of prices. If the price levels of a very large country are compared to the price levels of a small country using formula (3.17), then the quantity vector of the large country may totally overwhelm the influence of the quantity vector corresponding to the

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[25] Note that we have chosen the mean function $m(q_i^0,q_i^1)$ to be the same for each item i. We assume that m(a,b) has the following two properties: m(a,b) is a positive and continuous function, defined for all positive numbers a and b and m(a,a) = a for all a > 0.

[26] For more on symmetric means, see Diewert (1993c; 361).

[27] Walsh endorsed P_W as being the best index number formula: "We have seen reason to believe formula 6 better than formula 7. Perhaps formula 9 is the best of the rest, but between it and Nos. 6 and 8 it would be difficult to decide with assurance." C.M. Walsh (1921; 103). His formula 6 is P_W defined by (3.19) and his 9 is the Fisher ideal defined by (3.12) above. The *Walsh quantity index*, $Q_W(p^0,p^1,q^0,q^1)$ is defined as $P_W(q^0,q^1,p^0,p^1)$; i.e., the role of prices and quantities in definition (3.19) is interchanged. If we use the Walsh quantity index to deflate the value ratio, we obtain an implicit price index which is Walsh's formula 8.

small country.[28] In technical terms, the Edgeworth Marshall formula is not homogeneous of degree 0 in the components of both q^0 and q^1. To prevent this problem from occurring in the use of the pure price index $P_K(p^0,p^1,q^0,q^1)$ defined by (3.17), we ask that P_K satisfy the following *invariance to proportional changes in current quantities test*.[29]

(3.20) $P_K(p^0,p^1,q^0,\lambda q^1) = P_K(p^0,p^1,q^0,q^1)$ for all p^0,p^1,q^0,q^1 and all $\lambda > 0$.

The two tests, the time reversal test (3.13) and the invariance test (3.20), enable us to determine the precise functional form for the pure price index P_K defined by (3.17) above: the pure price index P_K must be the Walsh index P_W defined by (3.19).[30]

35. In order to be of practical use by statistical agencies, an index number formula must be able to be expressed as a function of the base period expenditure shares, s_i^0, the current period expenditure shares, s_i^1, and the n price ratios, p_i^1/p_i^0. The Walsh price index defined by (3.19) above can be rewritten in this format:

(3.21) $$\begin{aligned} P_W(p^0,p^1,q^0,q^1) &\equiv \sum_{i=1}^n p_i^1(q_i^0q_i^1)^{1/2} / \sum_{j=1}^n p_j^0(q_j^0q_j^1)^{1/2} \\ &= \sum_{i=1}^n [p_i^1/(p_i^0p_i^1)^{1/2}]\,(s_i^0s_i^1)^{1/2} / \sum_{j=1}^n [p_j^0/(p_j^0p_j^1)^{1/2}]\,(s_j^0s_j^1)^{1/2} \\ &= \sum_{i=1}^n (s_i^0s_i^1)^{1/2}\,[p_i^1/p_i^0]^{1/2} / \sum_{j=1}^n (s_j^0s_j^1)^{1/2}\,[p_j^0/p_j^1]^{1/2} . \end{aligned}$$

36. The approach to index number theory in this section was to consider averages of various fixed basket type price indices. Our first approach was to take an even handed average of the two primary fixed basket indices: the Laspeyres and Paasche price indices. These two primary indices are based on pricing out the baskets that pertain to the two periods (or locations) under consideration. Taking an average of them led to the Fisher ideal price index P_F defined by (3.12) above. Our second approach was to average the basket quantity weights and then price out this average basket at the prices pertaining to the two situations under consideration. This approach led to the Walsh price index P_W defined by (3.19) above Both of these indices can be written as a function of the base period expenditure shares, s_i^0, the current period expenditure shares, s_i^1, and the n price ratios, p_i^1/p_i^0. Assuming that the statistical agency has information on these three sets of variables, which index should be used? Experience with normal time series data has shown that these two indices will not differ substantially and thus it is a matter of indifference which of these indices is used in practice.[31] Both of these indices are examples of *superlative indices*, which will be defined in section H below. However, note that both of these indices treat the data pertaining to the two situations in a *symmetric* manner. Hill[32] commented on superlative price indices and the importance of a symmetric treatment of the data as follows:

[28] This is not likely to be a severe problem in the time series context where the change in quantity vectors going from one period to the next is small.
[29] This is the terminology used by Diewert (1992; 216). Vogt (1980) was the first to propose this test.
[30] See section 7 of Diewert (2001).
[31] Diewert (1978; 887-889) shows that these two indices will approximate each other to the second order around an equal price and quantity point. Thus for normal time series data where prices and quantities do not change much going from the base period to the current period, the indices will approximate each other quite closely. However, see the discussion in section H.6 below.
[32] See also Hill (1988).

"Thus economic theory suggests that, in general, a symmetric index that assigns equal weight to the two situations being compared is to be preferred to either the Laspeyres or Paasche indices on their own. The precise choice of superlative index—whether Fisher, Törnqvist or other superlative index—may be of only secondary importance as all the symmetric indices are likely to approximate each other, and the underlying theoretic index fairly closely, at least when the index number spread between the Laspeyres and Paasche is not very great." Peter Hill (1993; 384).

D. The Divisia index and discrete approximations to it

D.1 The Divisia price and quantity indices

3**7.** The second approach to index number theory relies on the assumption that price and quantity data change in a more or less continuous way.

38. Suppose that our price and quantity data on the n commodities in our chosen domain of definition can be regarded as continuous functions of (continuous) time, say $p_i(t)$ and $q_i(t)$ for i = 1,…,n. The value of consumer expenditure at time t is V(t) defined in the obvious way as:

(3.22) $V(t) \equiv \sum_{i=1}^{n} p_i(t)q_i(t)$.

Now suppose that the functions $p_i(t)$ and $q_i(t)$ are differentiable. Then we can differentiate both sides of (3.22) with respect to time to obtain:

(3.23) $V'(t) = \sum_{i=1}^{n} p_i'(t)q_i(t) + \sum_{i=1}^{n} p_i(t)q_i'(t)$.

Now divide both sides of (3.23) through by V(t) and using (3.22), we obtain the following equation:

(3.24) $$\begin{aligned} V'(t)/V(t) &= [\sum_{i=1}^{n} p_i'(t)q_i(t) + \sum_{i=1}^{n} p_i(t)q_i'(t)] / \sum_{j=1}^{n} p_j(t)q_j(t) \\ &= \sum_{i=1}^{n} [p_i'(t)/p_i(t)]\, s_i(t) + \sum_{i=1}^{n} [q_i'(t)/q_i(t)] s_i(t) \end{aligned}$$

where the time t expenditure share on commodity i, $s_i(t)$, is defined as:

(3.25) $s_i(t) \equiv p_i(t)q_i(t) / \sum_{m=1}^{n} p_m(t)q_m(t)$ for i = 1,2,…,n.

Now Divisia (1926; 39) argued as follows: *suppose* the aggregate value at time t, V(t), can be written as the product of a time t price level function, P(t) say, times a time t quantity level function, Q(t) say; i.e., we have:

(3.26) $V(t) = P(t)Q(t)$.

Suppose further that the functions P(t) and Q(t) are differentiable. Then differentiating (3.26) yields:

(3.27) $V'(t) = P'(t)Q(t) + P(t)Q'(t)$.

Dividing both sides of (3.27) by V(t) and using (3.26) leads to the following equation:

(3.28) $V'(t)/V(t) = [P'(t)/P(t)] + [Q'(t)/Q(t)]$.

Divisia compared the two expressions for the logarithmic value derivative, V′(t)/V(t), given by (3.24) and (3.28) and he simply *defined* the logarithmic rate of change of the *aggregate price level*, P′(t)/P(t), as the first set of terms on the right hand side of (3.24) and he simply *defined* the logarithmic rate of change of the *aggregate quantity level*, Q′(t)/Q(t), as the second set of terms on the right hand side of (3.24); i.e., he made the following definitions:

(3.29) $P'(t)/P(t) \equiv \sum_{i=1}^{n} s_i(t)\,[p_i'(t)/p_i(t)]$;
(3.30) $Q'(t)/Q(t) \equiv \sum_{i=1}^{n} s_i(t)\,[q_i'(t)/q_i(t)]$.

39. Definitions (3.29) and (3.30) are reasonable definitions for the proportional changes in the aggregate price and quantity (or quantity) levels, P(t) and Q(t). The problem with these definitions is that economic data are not collected in *continuous* time; they are collected in *discrete* time. In other words, even though transactions can be thought of as occurring in continuous time, no consumer records his or her purchases as they occur in continuous time; rather, purchases over a finite time period are cumulated and then recorded. A similar situation occurs for producers or sellers of commodities; firms cumulate their sales over discrete periods of time for accounting or analytical purposes. If we attempt to approximate continuous time by shorter and shorter discrete time intervals, we can expect empirical price and quantity data to become increasingly erratic since consumers only make purchases at discrete points of time (and producers or sellers of commodities only make sales at discrete points of time). However, it is still of some interest to approximate the continuous time price and quantity levels, P(t) and Q(t) defined implicitly by (3.29) and (3.30), by discrete time approximations. This can be done in two ways. We can either use methods of numerical approximation or make assumptions about the path taken by the functions $p_i(t)$ and $q_i(t)$ ($i = 1,\ldots,n$) through time. We will follow the first strategy in the following section. For discussions of the second strategy, see Vogt (1977) (1978), Van Ijzeren (1987; 8-12), Vogt and Barta (1997) and Balk (2000a).

40. There is a connection between the Divisia price and quantity levels, P(t) and Q(t), and the economic approach to index number theory. However, this connection is best made after we have covered the economic approach to index number theory. Since this material is rather technical, it has been relegated to Appendix 3.2.

D.2 Discrete approximations to the continuous time Divisia index

41. In order to make operational the continuous time Divisia price and quantity levels, P(t) and Q(t) defined by the differential equations (3.29) and (3.30), we have to convert to discrete time. Divisia (1926; 40) suggested a straightforward method for doing this conversion, which we now outline.

42. Define the following price and quantity (forward) differences:

(3.31) $\Delta P \equiv P(1) - P(0)$;
(3.32) $\Delta p_i \equiv p_i(1) - p_i(0)$; $i = 1,\ldots,n$.

Using the above definitions, we have:

(3.33) $P(1)/P(0) = [P(0) + \Delta P]/P(0)$ using (3.31)
$= 1 + [\Delta P/P(0)]$
$\approx 1 + [\sum_{i=1}^{n} \Delta p_i\, q_i(0)]/[\sum_{m=1}^{n} p_m(0)q_m(0)]$
using (3.29) when t = 0 and approximating $p_i'(0)$ by the difference Δp_i
$= [\sum_{i=1}^{n} \{p_i(0) + \Delta p_i\}\, q_i(0)]/[\sum_{m=1}^{n} p_m(0)q_m(0)]$
$= [\sum_{i=1}^{n} p_i(1)\, q_i(0)]/[\sum_{m=1}^{n} p_m(0)q_m(0)]$ using $p_n(1) = p_n(0) + \Delta p_n$
$= P_L(p^0,p^1,q^0,q^1)$

where we define $p^t \equiv [p_1(t),\ldots,p_n(t)]$ and $q^t \equiv [q_1(t),\ldots,q_n(t)]$ for t = 0,1. Thus, it can be seen that Divisia's discrete approximation to his continuous time price index is just our old friend, the Laspeyres price index, P_L defined above by (3.5).

43. But now we run into the problem noted by Frisch (1936; 8): instead of approximating the derivatives by the discrete (forward) differences defined by (3.31) and (3.32), we could use other approximations and obtain a wide variety of discrete time approximations. For example, instead of using forward differences and evaluating the index at time t = 0, we could use backward differences and evaluate the index at time t = 1. These backward differences are defined as:

(3.34) $\Delta_b p_i \equiv p_i(0) - p_i(1)$; $i = 1,\ldots,n$.

This use of backward differences leads to the following approximation for P(0)/P(1):

(3.35) $P(0)/P(1) = [P(1) + \Delta_b P]/P(1)$
$= 1 + [\Delta_b P/P(1)]$
$\approx 1 + [\sum_{i=1}^{n} \Delta_b p_i\, q_i(1)]/[\sum_{m=1}^{n} p_m(1)q_m(1)]$
using (3.29) when t = 1 and approximating $p_i'(1)$ by the difference $\Delta_b p_i$
$= [\sum_{i=1}^{n} \{p_i(1) + \Delta_b p_i\}\, q_i(1)]/[\sum_{m=1}^{n} p_m(1)q_m(1)]$
$= [\sum_{i=1}^{n} p_i(0)\, q_i(1)]/[\sum_{m=1}^{n} p_i(1)q_i(1)]$ using $p_i(0) = p_i(1) + \Delta_b p_i$
$= 1/P_P(p^0,p^1,q^0,q^1)$

where P_P is the Paasche index defined above by (3.6). Taking reciprocals of both sides of (3.35) leads to the following discrete approximation to P(1)/P(0):

(3.36) $P(1)/P(0) \approx P_P$.

44. Thus, as Frisch[33] noted, both the Paasche and Laspeyres indices can be regarded as (equally valid) approximations to the continuous time Divisia price index.[34] Since the Paasche and Laspeyres indices can differ considerably in some empirical applications, it can be seen that Divisia's idea is not all that helpful in determining a *unique* discrete time index number formula.[35] What is useful about the Divisia indices is the idea that as the discrete unit of time gets smaller, discrete approximations to the Divisia indices can approach meaningful economic indices under certain conditions.

E. Fixed base versus chain indices

45. In this section[36], we discuss the merits of using the chain system for constructing price indices in the time series context versus using the fixed base system.[37]

46. The chain system[38] measures the change in prices going from one period to a subsequent period using a bilateral index number formula involving the prices and quantities pertaining to the two adjacent periods. These one period rates of change (the links in the chain) are then cumulated to yield the relative levels of prices over the entire period under consideration. Thus if the bilateral price index is P, the chain system generates the following pattern of price levels for the first three periods:

$$(3.37)\quad 1,\ P(p^0,p^1,q^0,q^1),\ P(p^0,p^1,q^0,q^1)P(p^1,p^2,q^1,q^2).$$

47. On the other hand, the fixed base system of price levels using the same bilateral index number formula P simply computes the level of prices in period t relative to the base period 0 as $P(p^0,p^t,q^0,q^t)$. Thus the fixed base pattern of price levels for periods 0,1 and 2 is:

$$(3.38)\quad 1,\ P(p^0,p^1,q^0,q^1),\ P(p^0,p^2,q^0,q^2).$$

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33 "As the elementary formula of the chaining, we may get Laspeyre's or Paasche's or Edgeworth's or nearly any other formula, according as we choose the approximation principle for the steps of the numerical integration." Ragnar Frisch (1936; 8).

34 Diewert (1980; 444) also obtained the Paasche and Laspeyres approximations to the Divisia index using a somewhat different approximation argument. He also showed how several other popular discrete time index number formulae could be regarded as approximations to the continuous time Divisia index.

35 Trivedi (1981) systematically examined the problems involved in finding a "best" discrete time approximation to the Divisia indices using the techniques of numerical analysis. However, these numerical analysis techniques depend on the assumption that the "true" continuous time micro price functions, $p_i(t)$, can be adequately represented by a polynomial approximation. Thus we are led to the conclusion that the "best" discrete time approximation to the Divisia index depends on assumptions that are difficult to verify.

36 This section is largely based on Hill (1988) (1993; 385-390).

37 The results in Appendix 3.2 provide some theoretical support for the use of chain indices in that it is shown that under certain conditions, the Divisia index will equal an economic index. Hence any discrete approximation to the Divisia index will approach the economic index as the time period gets shorter. Thus under certain conditions, chain indices will approach an underlying economic index.

38 The chain principle was introduced independently into the economics literature by Lehr (1885; 45-46) and Marshall (1887; 373). Both authors observed that the chain system would mitigate the difficulties due to the introduction of new commodities into the economy, a point also mentioned by Hill (1993; 388). Fisher (1911; 203) introduced the term "chain system".

48. Note that in both the chain system and the fixed base system of price levels defined by (3.37) and (3.38) above, we have set the base period price level equal to 1. The usual practice in statistical agencies is to set the base period price level equal to 100. If this is done, then it is necessary to multiply each of the numbers in (3.37) and (3.38) by 100.

49. Due to the difficulties involved in obtaining current period information on quantities (or equivalently, on expenditures), many statistical agencies base their Consumer Price Index on the use of the Laspeyres formula (3.5) and the fixed base system. Therefore, it is of some interest to look at some of the possible problems associated with the use of fixed base Laspeyres indices.

50. The main problem with the use of fixed base Laspeyres indices is that the period 0 fixed basket of commodities that is being priced out in period t can often be quite different from the period t basket. Thus if there are systematic *trends* in at least some of the prices and quantities[39] in the index basket, the fixed base Laspeyres price index $P_L(p^0,p^t,q^0,q^t)$ can be quite different from the corresponding fixed base Paasche price index, $P_P(p^0,p^t,q^0,q^t)$.[40] This means that both indices are likely to be an inadequate representation of the movement in average prices over the time period under consideration.

51. The fixed base Laspeyres quantity index cannot be used forever: eventually, the base period quantities q^0 are so far removed from the current period quantities q^t that the base must be changed. Chaining is merely the limiting case where the base is changed each period.[41]

52. The main advantage of the chain system is that under normal conditions, chaining will reduce the spread between the Paasche and Laspeyres indices.[42] These two indices each provide an asymmetric perspective on the amount of price change that has occurred between the two periods under consideration and we would expect that a single point estimate of the aggregate price change should lie between these two estimates. Thus the use of either a chained Paasche or Laspeyres index will usually lead to a smaller difference between the two and hence to estimates that are closer to the "truth".

53. Hill (1993; 388), drawing on the earlier research of Szulc (1983) and Hill (1988; 136-137), noted that it is not appropriate to use the chain system when prices oscillate (or "bounce" to use Szulc's (1983; 548) term). This phenomenon can occur in the context of

[39] Examples of rapidly downward trending prices and upward trending quantities are computers, electronic equipment of all types, internet access and telecommunication charges.

[40] Note that $P_L(p^0,p^t,q^0,q^t)$ will equal $P_P(p^0,p^t,q^0,q^t)$ if *either* the two quantity vectors q^0 and q^t are proportional *or* the two price vectors p^0 and p^t are proportional. Thus in order to obtain a difference between the Paasche and Laspeyres indices, we require relative change in *both* prices and quantities.

[41] Regular seasonal fluctuations can cause monthly or quarterly data to "bounce" using the term due to Szulc (1983) and chaining bouncing data can lead to a considerable amount of index "drift"; i.e., if after 12 months, prices and quantities return to their levels of a year earlier, then a chained monthly index will usually not return to unity. Hence, we do not recommend the use of chained indices for monthly or quarterly data.

[42] See Diewert (1978; 895) and Hill (1988) (1993; 387-388).

regular seasonal fluctuations or in the context of price wars. However, in the context of roughly monotonically changing prices and quantities, Hill (1993; 389) recommended the use of chained symmetrically weighted indices (see section C above). The Fisher and Walsh indices are examples of symmetrically weighted indices.

54. It is of some interest to determine if there are index number formulae that give the same answer when either the fixed base or chain system is used. Comparing the sequence of chain indices defined by (3.38) above to the corresponding fixed base indices, it can be seen that we will obtain the same answer in all three periods if the index number formula P satisfies the following functional equation for all price and quantity vectors:

$$(3.39)\quad P(p^0,p^2,q^0,q^2) = P(p^0,p^1,q^0,q^1)P(p^1,p^2,q^1,q^2).$$

If an index number formula P satisfies (3.39), then we say that P satisfies the *circularity test*.[43]

55. If we assume that the index number formula P satisfies certain properties or tests in addition to the circularity test above[44], then Funke, Hacker and Voeller (1979) show that P must have the following functional form due originally to Konüs and Byushgens[45] (1926; 163-166):[46]

$$(3.40)\quad P_{KB}(p^0,p^1,q^0,q^1) \equiv \prod_{i=1}^{n} [p_i^1/p_i^0]^{\alpha_i}$$

where the n constants α_i satisfy the following restrictions:

$$(3.41)\quad \sum_{i=1}^{n} \alpha_i = 1 \text{ and } \alpha_i > 0 \text{ for } i = 1,\ldots,n.$$

Thus under very weak regularity conditions, the only price index satisfying the circularity test is a weighted geometric average of all the individual price ratios, the weights being constant through time.

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[43] The test name is due to Fisher (1922; 413) and the concept was originally due to Westergaard (1890; 218-219).

[44] The additional tests are: (i) positivity and continuity of $P(p^0,p^1,q^0,q^1)$ for all strictly positive price and quantity vectors p^0,p^1,q^0,q^1; (ii) the identity test; (iii) the commensurability test; (iv) $P(p^0,p^1,q^0,q^1)$ is positively homogeneous of degree one in the components of p^1 and (v)) $P(p^0,p^1,q^0,q^1)$ is positively homogeneous of degree zero in the components of q^1. These tests will be explained in section E below.

[45] Konüs and Byushgens show that the index defined by (3.40) is exact for Cobb-Douglas (1928) preferences; see also Pollak (1989; 23). The concept of an exact index number formula will be explained in section H below.

[46] This result can be derived using results in Eichhorn (1978; 167-168) and Vogt and Barta (1997; 47). A simple proof can be found in Balk (1995). This result vindicates Irving Fisher's (1922; 274) intuition who asserted that "the only formulae which conform perfectly to the circular test are index numbers which have *constant weights*…" Fisher (1922; 275) went on to assert: "But, clearly, constant weighting is not theoretically correct. If we compare 1913 with 1914, we need one set of weights; if we compare 1913 with 1915, we need, theoretically at least, another set of weights. … Similarly, turning from time to space, an index number for comparing the United States and England requires one set of weights, and an index number for comparing the United States and France requires, theoretically at least, another."

56. An interesting special case of the family of indices defined by (3.40) occurs when the weights α_i are all equal. In this case, P_{KB} reduces to the Jevons (1865) index:

$$(3.42)\quad P_J(p^0,p^1,q^0,q^1) \equiv \prod_{i=1}^{n} [p_i^1/p_i^0]^{1/n} .$$

57. The problem with the indices defined by Konüs and Byushgens and Voeller and Jevons is that the individual price ratios, p_i^1/p_i^0, have weights (either α_i or $1/n$) that are *independent* of the economic importance of commodity i in the two periods under consideration. Put another way, these price weights are independent of the quantities of commodity i consumed or the expenditures on commodity i during the two periods. Hence, these indices are not really suitable for use by statistical agencies at higher levels of aggregation when expenditure share information is available.

58. The above results indicate that it is not useful to ask that the price index P satisfy the circularity test *exactly*. However, it is of some interest to find index number formulae that satisfy the circularity test to some degree of *approximation* since the use of such an index number formula will lead to measures of aggregate price change that are more or less the same no matter whether we use the chain or fixed base systems. Irving Fisher (1922; 284) found that deviations from circularity using his data set and the Fisher ideal price index P_F defined by (3.12) above were quite small. This relatively high degree of correspondence between fixed base and chain indices has been found to hold for other symmetrically weighted formulae like the Walsh index P_W defined by (3.19) above.[47] Thus in most time series applications of index number theory where the base year in fixed base indices is changed every 5 years or so, it will not matter very much whether the statistical agency uses a fixed base price index or a chain index, provided that a symmetrically weighted formula is used.[48] This of course depends on the length of the time series considered and the degree of variation in the prices and quantities as we go from period to period. The more prices and quantities are subject to large fluctuations (rather than smooth trends), the less the correspondence.[49]

59. It is possible to give a theoretical explanation for the approximate satisfaction of the circularity test for symmetrically weighted index number formulae. Another symmetrically weighted formula is the Törnqvist index P_T.[50] The natural logarithm of this index is defined as follows:

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[47] See for example Diewert (1978; 894).

[48] More specifically, most superlative indices (which are symmetrically weighted) will satisfy the circularity test to a high degree of approximation in the time series context. See section H below for the definition of a superlative index. It is worth stressing that fixed base Paasche and Laspeyres indices are very likely to diverge considerably over a 5 year period if computers (or any other commodity which has price and quantity trends that are quite different from the trends in the other commodities) are included in the value aggregate under consideration. See Appendix 3.3 below for some “empirical” evidence on this topic.

[49] Again, see Szulc (1983) and Hill (1988).

[50] This formula was explicitly defined in Törnqvist and Törnqvist (1937).

(3.43) $\ln P_T(p^0,p^1,q^0,q^1) \equiv \sum_{i=1}^n (1/2)(s_i^0 + s_i^1) \ln (p_i^1/p_i^0)$

where the period t expenditure shares s_i^t are defined by (3.7) above. Alterman, Diewert and Feenstra (1999; 61) show that if the logarithmic price ratios $\ln (p_i^t/p_i^{t-1})$ trend linearly with time t and the expenditure shares s_i^t also trend linearly with time, then the Törnqvist index P_T will satisfy the circularity test exactly.[51] Since many economic time series on prices and quantities satisfy these assumptions approximately, then the Törnqvist index P_T will satisfy the circularity test approximately. As we shall see in section H.6 below, the Törnqvist index generally closely approximates the symmetrically weighted Fisher and Walsh indices, so that for many economic time series, all three of these symmetrically weighted indices will satisfy the circularity test to a high enough degree of approximation so that it will not matter whether we use the fixed base or chain principle.

60. We have already introduced various properties, axioms or tests that an index number formula could satisfy in this chapter. In the following section, we study the test approach to index number theory in a more systematic manner.

F. The axiomatic approach to price indices

F.1 Bilateral indices and some early tests

61. In this section, our goal will be to assume that the bilateral price index formula, $P(p^0,p^1,q^0,q^1)$, satisfies a sufficient number of "reasonable" tests or properties so that the functional form for P is determined.[52] The word "bilateral"[53] refers to the assumption that the function P depends only on the data pertaining to the two situations or periods being compared; i.e., P is regarded as a function of the two sets of price and quantity vectors, p^0,p^1,q^0,q^1, that are to be aggregated into a single number that summarizes the overall change in the n price ratios, $p_1^1/p_1^0,\ldots, p_n^1/p_n^0$.

62. We will take the perspective outlined in section B.1 above; i.e., along with the price index $P(p^0,p^1,q^0,q^1)$, there is a companion quantity index $Q(p^0,p^1,q^0,q^1)$ such that the product of these two indices equals the value ratio between the two periods. Thus, throughout this section, we assume that P and Q satisfy the product test, (3.3) above. This means that as soon as the functional form for the price index P is determined, then (3.3) can be used to determine the functional form for the quantity index Q. However, a further advantage of assuming that the product test holds is that we can assume that the quantity index Q satisfies a "reasonable" property and then use (3.3) to translate this test on the quantity index into a corresponding test on the price index P.[54]

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[51] This exactness result can be extended to cover the case when there are monthly proportional variations in prices and the expenditure shares have constant seasonal effects in addition to linear trends; see Alterman, Diewert and Feenstra (1999; 65).
[52] Much of the material in this section is drawn from section sections 2 and 3 of Diewert (1992). For a survey of the axiomatic approach see Balk (1995).
[53] Multilateral index number theory refers to the situation where there are more than two situations whose prices and quantities need to be aggregated.
[54] This observation was first made by Fisher (1911; 400-406). Vogt (1980) also pursued this idea.

63. If n = 1, so that there is only one price and quantity to be aggregated, then a natural candidate for P is p_1^1/p_1^0 , the single price ratio, and a natural candidate for Q is q_1^1/q_1^0 , the single quantity ratio. When the number of commodities or items to be aggregated is greater than 1, then what index number theorists have done over the years is propose properties or tests that the price index P should satisfy. These properties are generally multi-dimensional analogues to the one good price index formula, p_1^1/p_1^0. Below, we list twenty tests that turn out to characterize the Fisher ideal price index.

64. We shall assume that every component of each price and quantity vector is positive; i.e., $p^t >> 0_n$ and $q^t >> 0_n$ [55] for t = 0,1. If we want to set $q^0 = q^1$, we call the common quantity vector q; if we want to set $p^0 = p^1$, we call the common price vector p.

65. Our first two tests are not very controversial and so we will not discuss them.

T1: *Positivity* [56]: $P(p^0,p^1,q^0,q^1) > 0$.

T2: *Continuity* [57]: $P(p^0,p^1,q^0,q^1)$ is a continuous function of its arguments.

62. Our next two tests are somewhat more controversial.

T3: *Identity or Constant Prices Test* [58]: $P(p,p,q^0,q^1) = 1$.

That is, if the price of every good is identical during the two periods, then the price index should equal unity, no matter what the quantity vectors are. The controversial part of this test is that the two quantity vectors are allowed to be different in the above test.[59]

T4: *Fixed Basket or Constant Quantities Test* [60]: $P(p^0,p^1,q,q) = \sum_{i=1}^n p_i^1 q_i / \sum_{i=1}^n p_i^0 q_i$.

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[55] Notation: $q >> 0_n$ means that each component of the vector q is positive; $q \geq 0_n$ means each component of q is nonnegative and $q > 0_n$ means $q \geq 0_n$ and $q \neq 0_n$.
[56] Eichhorn and Voeller (1976, 23) suggested this test.
[57] Fisher (1922; 207-215) informally suggested the essence of this test.
[58] Laspeyres (1871; 308), Walsh (1901; 308) and Eichhorn and Voeller (1976; 24) have all suggested this test. Laspeyres came up with this test or property to discredit the ratio of unit values index of Drobisch (1871a), which does not satisfy this test. This test is also a special case of Fisher's (1911; 409-410) price proportionality test.
[59] Usually, economists assume that given a price vector p, the corresponding quantity vector q is uniquely determined. Here, we have the same price vector but the corresponding quantity vectors are allowed to be different.
[60] The origins of this test go back at least two hundred years to the Massachusetts legislature which used a constant basket of goods to index the pay of Massachusetts soldiers fighting in the American Revolution; see Willard Fisher (1913). Other researchers who have suggested the test over the years include: Lowe (1823, Appendix, 95), Scrope (1833, 406), Jevons (1865), Sidgwick (1883, 67-68), Edgeworth (1925, 215) originally published in 1887, Marshall (1887, 363), Pierson (1895, 332), Walsh (1901, 540) (1921; 544), and Bowley (1901, 227). Vogt and Barta (1997; 49) correctly observe that this test is a special case of Fisher's (1911; 411) proportionality test for quantity indexes which Fisher (1911; 405) translated into a test for the price index using the product test (3.3).

That is, if quantities are constant during the two periods so that $q^0 = q^1 \equiv q$, then the price index should equal the expenditure on the constant basket in period 1, $\sum_{i=1}^n p_i^1 q_i$, divided by the expenditure on the basket in period 0, $\sum_{i=1}^n p_i^0 q_i$.

66. If the price index P satisfies Test T4 and P and Q jointly satisfy the product test, (3.3) above, then it is easy to show[61] that Q must satisfy the identity test $Q(p^0,p^1,q,q) = 1$ for all strictly positive vectors p^0,p^1,q. This *constant quantities test* for Q is also somewhat controversial since p^0 and p^1 are allowed to be different.

67. We turn now to some homogeneity tests for P.

F.2 Homogeneity tests

68. The following four tests restrict the behavior of the price index P as the scale of any one of the four vectors p^0,p^1,q^0,q^1 changes.

T5: *Proportionality in Current Prices* [62]: $P(p^0,\lambda p^1,q^0,q^1) = \lambda P(p^0,p^1,q^0,q^1)$ for $\lambda > 0$.

That is, if all period 1 prices are multiplied by the positive number λ, then the new price index is λ times the old price index. Put another way, the price index function $P(p^0,p^1,q^0,q^1)$ is (positively) homogeneous of degree one in the components of the period 1 price vector p^1. Most index number theorists regard this property as a very fundamental one that the index number formula should satisfy.

69. Walsh (1901) and Fisher (1911; 418) (1922; 420) proposed the related proportionality test $P(p,\lambda p,q^0,q^1) = \lambda$. This last test is a combination of T3 and T5; in fact Walsh (1901, 385) noted that this last test implies the identity test, T3.

70. In our next test, instead of multiplying all period 1 prices by the same number, we multiply all period 0 prices by the number λ.

T6: *Inverse Proportionality in Base Period Prices* [63]: $P(\lambda p^0,p^1,q^0,q^1) = \lambda^{-1}P(p^0,p^1,q^0,q^1)$ for $\lambda > 0$.

That is, if all period 0 prices are multiplied by the positive number λ, then the new price index is $1/\lambda$ times the old price index. Put another way, the price index function $P(p^0,p^1,q^0,q^1)$ is (positively) homogeneous of degree minus one in the components of the period 0 price vector p^0.

71. The following two homogeneity tests can also be regarded as invariance tests.

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[61] See Vogt (1980; 70).
[62] This test was proposed by Walsh (1901, 385), Eichhorn and Voeller (1976, 24) and Vogt (1980, 68).
[63] Eichhorn and Voeller (1976; 28) suggested this test.

T7: *Invariance to Proportional Changes in Current Quantities*: $P(p^0,p^1,q^0,\lambda q^1) = P(p^0,p^1,q^0,q^1)$ for all $\lambda > 0$.

That is, if current period quantities are all multiplied by the number λ, then the price index remains unchanged. Put another way, the price index function $P(p^0,p^1,q^0,q^1)$ is (positively) homogeneous of degree zero in the components of the period 1 quantity vector q^1. Vogt (1980, 70) was the first to propose this test[64] and his derivation of the test is of some interest. Suppose the quantity index Q satisfies the quantity analogue to the price test T5; i.e., suppose Q satisfies $Q(p^0,p^1,q^0,\lambda q^1) = \lambda Q(p^0,p^1,q^0,q^1)$ for $\lambda > 0$. Then using the product test (3.3), we see that P must satisfy T7.

T8: *Invariance to Proportional Changes in Base Quantities* [65]: $P(p^0,p^1,\lambda q^0,q^1) = P(p^0,p^1,q^0,q^1)$ for all $\lambda > 0$.

That is, if base period quantities are all multiplied by the number λ, then the price index remains unchanged. Put another way, the price index function $P(p^0,p^1,q^0,q^1)$ is (positively) homogeneous of degree zero in the components of the period 0 quantity vector q^0. If the quantity index Q satisfies the following counterpart to T8: $Q(p^0,p^1,\lambda q^0,q^1) = \lambda^{-1}Q(p^0,p^1,q^0,q^1)$ for all $\lambda > 0$, then using (3.3), the corresponding price index P must satisfy T8. This argument provides some additional justification for assuming the validity of T8 for the price index function P.

72. T7 and T8 together impose the property that the price index P does not depend on the *absolute* magnitudes of the quantity vectors q^0 and q^1.

F.3 Invariance and symmetry tests

73. The next five tests are invariance or symmetry tests. Fisher (1922; 62-63, 458-460) and Walsh (1921; 542) seem to have been the first researchers to appreciate the significance of these kinds of tests. Fisher (1922, 62-63) spoke of fairness but it is clear that he had symmetry properties in mind. It is perhaps unfortunate that he did not realize that there were more symmetry and invariance properties than the ones he proposed; if he had realized this, it is likely that he would have been able to provide an axiomatic characterization for his ideal price index, as will be done in section F.6 below. Our first invariance test is that the price index should remain unchanged if the *ordering* of the commodities is changed:

T9: *Commodity Reversal Test* (or invariance to changes in the ordering of commodities):

$$P(p^{0*},p^{1*},q^{0*},q^{1*}) = P(p^0,p^1,q^0,q^1)$$

where p^{t*} denotes a permutation of the components of the vector p^t and q^{t*} denotes the same permutation of the components of q^t for $t = 0,1$. This test is due to Irving Fisher

[64] Fisher (1911; 405) proposed the related test $P(p^0,p^1,q^0,\lambda q^0) = P(p^0,p^1,q^0,q^0) = \sum_{i=1}^n p_i^1 q_i^0 / \sum_{i=1}^n p_i^0 q_i^0$.
[65] This test was proposed by Diewert (1992; 216).

(1922), and it is one of his three famous reversal tests. The other two are the time reversal test and the factor reversal test which will be considered below.

T10: *Invariance to Changes in the Units of Measurement* (commensurability test):

$$P(\alpha_1 p_1^0,...,\alpha_n p_n^0;\ \alpha_1 p_1^1,...,\alpha_n p_n^1;\ \alpha_1^{-1} q_1^0,...,\alpha_n^{-1} q_n^0;\ \alpha_1^{-1} q_1^1,...,\alpha_n^{-1} q_n^1) =$$
$$P(p_1^0,...,p_n^0;\ p_1^1,...,p_n^1;\ q_1^0,...,q_n^0;\ q_1^1,...,q_n^1) \text{ for all } \alpha_1 > 0, \ldots, \alpha_n > 0.$$

That is, the price index does not change if the units of measurement for each commodity are changed. The concept of this test was due to Jevons (1884; 23) and the Dutch economist Pierson (1896; 131), who criticized several index number formula for not satisfying this fundamental test. Fisher (1911; 411) first called this test *the change of units test* and later, Fisher (1922; 420) called it the *commensurability test*.

T11: *Time Reversal Test*: $P(p^0,p^1,q^0,q^1) = 1/P(p^1,p^0,q^1,q^0)$.

That is, if the data for periods 0 and 1 are interchanged, then the resulting price index should equal the reciprocal of the original price index. Obviously, in the one good case when the price index is simply the single price ratio; this test is satisfied (as are all of the other tests listed in this section). When the number of goods is greater than one, many commonly used price indices fail this test; e.g., the Laspeyres (1871) price index, P_L defined earlier by (3.5), and the Paasche (1874) price index, P_P defined earlier by (3.6), both fail this fundamental test. The concept of the test was due to Pierson (1896; 128), who was so upset with the fact that many of the commonly used index number formulae did not satisfy this test, that he proposed that the entire concept of an index number should be abandoned. More formal statements of the test were made by Walsh (1901; 368) (1921; 541) and Fisher (1911; 534) (1922; 64).

74. Our next two tests are more controversial, since they are not necessarily consistent with the economic approach to index number theory. However, these tests are quite consistent with the weighted stochastic approach to index number theory to be discussed later in this chapter.

T12: *Quantity Reversal Test* (quantity weights symmetry test): $P(p^0,p^1,q^0,q^1) = P(p^0,p^1,q^1,q^0)$.

That is, if the quantity vectors for the two periods are interchanged, then the price index remains invariant. This property means that if quantities are used to weight the prices in the index number formula, then the period 0 quantities q^0 and the period 1 quantities q^1 must enter the formula in a symmetric or even handed manner. Funke and Voeller (1978; 3) introduced this test; they called it the *weight property*.

75. The next test is the analogue to T12 applied to quantity indices:

T13: *Price Reversal Test* (price weights symmetry test)[66]:

[66] This test was proposed by Diewert (1992;218).

$\{\sum_{i=1}^{n} p_i^1 q_i^1 / \sum_{i=1}^{n} p_i^0 q_i^0\}/P(p^0,p^1,q^0,q^1) = \{\sum_{i=1}^{n} p_i^0 q_i^1 / \sum_{i=1}^{n} p_i^1 q_i^0\}/P(p^1,p^0,q^0,q^1)$.

Thus if we use (3.4) to define the quantity index Q in terms of the price index P, then it can be seen that T13 is equivalent to the following property for the associated quantity index Q:

(344) $Q(p^0,p^1,q^0,q^1) = Q(p^1,p^0,q^0,q^1)$.

That is, if the price vectors for the two periods are interchanged, then the quantity index remains invariant. Thus if prices for the same good in the two periods are used to weight quantities in the construction of the quantity index, then property T13 implies that these prices enter the quantity index in a symmetric manner.

F.4 Mean value tests

76. The next three tests are mean value tests.

T14: *Mean Value Test for Prices* [67]:
$\min_i (p_i^1/p_i^0 : i=1,...,n) \le P(p^0,p^1,q^0,q^1) \le \max_i (p_i^1/p_i^0 : i = 1,...,n)$.

That is, the price index lies between the minimum price ratio and the maximum price ratio. Since the price index is supposed to be some sort of an average of the n price ratios, p_i^1/p_i^0, it seems essential that the price index P satisfy this test.

77. The next test is the analogue to T14 applied to quantity indices:

T15: *Mean Value Test for Quantities* [68]:
$\min_i (q_i^1/q_i^0 : i=1,...,n) \le \{V^1/V^0\}/ P(p^0,p^1,q^0,q^1) \le \max_i (q_i^1/q_i^0 : i = 1,...,n)$

where V^t is the period t value for the aggregate defined by (3.2) above. Using (3.4) to define the quantity index Q in terms of the price index P, we see that T15 is equivalent to the following property for the associated quantity index Q:

(3.45) $\min_i (q_i^1/q_i^0 : i=1,...,n) \le Q(p^0,p^1,q^0,q^1) \le \max_i (q_i^1/q_i^0 : i = 1,...,n)$.

That is, the implicit quantity index Q defined by P lies between the minimum and maximum rates of growth q_i^1/q_i^0 of the individual quantities.

78. In section C.1, we argued that it was very reasonable to take an average of the Laspeyres and Paasche price indices as a single “best” measure of overall price change. This point of view can be turned into a test:

T16: *Paasche and Laspeyres Bounding Test* [69]: The price index P lies between the Laspeyres and Paasche indices, P_L and P_P, defined by (3.5) and (3.6) above.

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[67] This test seems to have been first proposed by Eichhorn and Voeller (1976; 10).
[68] This test was proposed by Diewert (1992; 219).

We could propose a test where the implicit quantity index Q that corresponds to P via (3.4) is to lie between the Laspeyres and Paasche quantity indices, Q_L and Q_P, defined by (3.10) and (3.11) above. However, the resulting test turns out to be equivalent to test T16.

F.5 Monotonicity tests

79. Our final four tests are monotonicity tests; i.e., how should the price index $P(p^0,p^1,q^0,q^1)$ change as any component of the two price vectors p^0 and p^1 increases or as any component of the two quantity vectors q^0 and q^1 increases.

T17: *Monotonicity in Current Prices*: $P(p^0,p^1,q^0,q^1) < P(p^0,p^2,q^0,q^1)$ if $p^1 < p^2$.

That is, if some period 1 price increases, then the price index must increase, so that $P(p^0,p^1,q^0,q^1)$ is increasing in the components of p^1. This property was proposed by Eichhorn and Voeller (1976; 23) and it is a very reasonable property for a price index to satisfy.

T18: *Monotonicity in Base Prices*: $P(p^0,p^1,q^0,q^1) > P(p^2,p^1,q^0,q^1)$ if $p^0 < p^2$.

That is, if any period 0 price increases, then the price index must decrease, so that $P(p^0,p^1,q^0,q^1)$ is decreasing in the components of p^0 . This very reasonable property was also proposed by Eichhorn and Voeller (1976; 23).

T19: *Monotonicity in Current Quantities*: if $q^1 < q^2$, then
$\{\sum_{i=1}^n p_i^1 q_i^1/ \sum_{i=1}^n p_i^0 q_i^0\}/P(p^0,p^1,q^0,q^1) < \{\sum_{i=1}^n p_i^1 q_i^2/ \sum_{i=1}^n p_i^0 q_i^0\}/P(p^0,p^1,q^0,q^2)$.

T20: *Monotonicity in Base Quantities*: if $q^0 < q^2$, then
$\{\sum_{i=1}^n p_i^1 q_i^1/ \sum_{i=1}^n p_i^0 q_i^0\}/P(p^0,p^1,q^0,q^1) > \{\sum_{i=1}^n p_i^1 q_i^1/ \sum_{i=1}^n p_i^0 q_i^2\}/P(p^0,p^1,q^0,q^2)$.

If we define the implicit quantity index Q that corresponds to P using (3.4), we find that T19 translates into the following inequality involving Q:

(3.46) $Q(p^0,p^1,q^0,q^1) < Q(p^0,p^1,q^0,q^2)$ if $q^1 < q^2$.

That is, if any period 1 quantity increases, then the implicit quantity index Q that corresponds to the price index P must increase. Similarly, we find that T20 translates into:

(3.47) $Q(p^0,p^1,q^0,q^1) > Q(p^0,p^1,q^2,q^1)$ if $q^0 < q^2$.

That is, if any period 0 quantity increases, then the implicit quantity index Q must decrease. Tests T19 and T20 are due to Vogt (1980, 70).

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[69] Bowley (1901; 227) and Fisher (1922; 403) both endorsed this property for a price index.

80. This concludes our listing of tests. In the next section, we ask whether any index number formula $P(p^0,p^1,q^0,q^1)$ exists that can satisfy all twenty tests.

F.6 The Fisher ideal index and the test approach

81. It can be shown that the only index number formula $P(p^0,p^1,q^0,q^1)$ which satisfies tests T1 - T20 is the Fisher ideal price index P_F defined above by (3.12)[70]. To prove this assertion, it is relatively straightforward to show that the Fisher index satisfies all 20 tests.

82. The more difficult part of the proof is to show that it is the *only* index number formula which satisfies these tests. This part of the proof follows from the fact that if P satisfies the positivity test T1 and the three reversal tests, T11-T13, then P must equal P_F. To see this, rearrange the terms in the statement of test T13 into the following equation:

$$(3.48)\ \{\sum_{i=1}^n p_i^1 q_i^1 / \sum_{i=1}^n p_i^0 q_i^0\}/\{\sum_{i=1}^n p_i^0 q_i^1 / \sum_{i=1}^n p_i^1 q_i^0\}$$
$$= P(p^0,p^1,q^0,q^1) / P(p^1,p^0,q^0,q^1)$$
$$= P(p^0,p^1,q^0,q^1) / P(p^1,p^0,q^1,q^0) \quad \text{using T12, the quantity reversal test}$$
$$= P(p^0,p^1,q^0,q^1)\ P(p^0,p^1,q^0,q^1) \quad \text{using T11, the time reversal test.}$$

Now take positive square roots on both sides of (3.35) and we see that the left hand side of the equation is the Fisher index $P_F(p^0,p^1,q^0,q^1)$ defined by (3.12) and the right hand side is $P(p^0,p^1,q^0,q^1)$. Thus if P satisfies T1, T11, T12 and T13, it must equal the Fisher ideal index P_F.

83. The quantity index that corresponds to the Fisher price index using the product test (3.3) is Q_F , the Fisher quantity index, defined by (3.14).

84. It turns out that P satisfies yet another test, T21, which was Irving Fisher's (1921; 534) (1922; 72-81) third reversal test (the other two being T9 and T11):

T21: *Factor Reversal Test* (functional form symmetry test):

$$P(p^0,p^1,q^0,q^1)\ P(q^0,q^1,p^0,p^1) = \sum_{i=1}^n p_i^1 q_i^1 / \sum_{i=1}^n p_i^0 q_i^0 .$$

A justification for this test is the following one: if $P(p^0,p^1,q^0,q^1)$ is a good functional form for the price index, then if we reverse the roles of prices and quantities, $P(q^0,q^1,p^0,p^1)$ ought to be a good functional form for a quantity index (which seems to be a correct argument) and thus the product of the price index $P(p^0,p^1,q^0,q^1)$ and the quantity index $Q(p^0,p^1,q^0,q^1) = P(q^0,q^1,p^0,p^1)$ ought to equal the value ratio, V^1 / V^0 . The second part of this argument does not seem to be valid and thus many researchers over the years have objected to the factor reversal test. However, if one is willing to embrace T21 as a basic test, Funke and Voeller (1978; 180) showed that the only index number function $P(p^0,p^1,q^0,q^1)$ which satisfies T1 (positivity), T11 (time reversal test), T12 (quantity

[70] See Diewert (1992; 221).

reversal test) and T21 (factor reversal test) is the Fisher ideal index P_F defined by (3.12). Thus the price reversal test T13 can be replaced by the factor reversal test in order to obtain a minimal set of four tests that lead to the Fisher price index.[71]

F.7 The test performance of other indices

85. The Fisher price index P satisfies all 20 of the tests listed in sections F.1-F.5 above. Which tests do other commonly used price indices satisfy? Recall the Laspeyres index P_L defined by (3.5), the Paasche index P_P defined by (3.6), the Walsh index P_W defined by (3.19) and the Törnqvist index P_T defined by (3.43).

86. Straightforward computations show that the Paasche and Laspeyres price indices, P_L and P_P, fail only the three reversal tests, T11, T12 and T13. Since the quantity and price reversal tests, T12 and T13, are somewhat controversial and hence can be discounted, the test performance of P_L and P_P seems at first sight to be quite good. However, the failure of the time reversal test, T11, is a severe limitation associated with the use of these indices.

87. The Walsh price index, P_W, fails four tests: T13, the price reversal test; T16, the Paasche and Laspeyres bounding test; T19, the monotonicity in current quantities test; and T20, the monotonicity in base quantities test.

88. Finally, the Törnqvist price index P_T fails nine tests: T4 (the fixed basket test), the quantity and price reversal tests T12 and T13, T15 (the mean value test for quantities), T16 (the Paasche and Laspeyres bounding test) and the 4 monotonicity tests T17 to T20. Thus the Törnqvist index is subject to a rather high failure rate.[72]

89. The conclusion we draw from the above results is that from the viewpoint of the test approach to index numbers, the Fisher ideal price index P_F appears to be "best" since it satisfies all 20 tests.[73] The Paasche and Laspeyres indices are next best if we treat each test as being equally important. However, both of these indices fail the very important time reversal test. The remaining two indices, the Walsh and Törnqvist price indices, both satisfy the time reversal test but the Walsh index emerges as being "better" since it passes 16 of our 20 tests whereas the Törnqvist only satisfies 11 tests.

F.8 The additivity test

[71] Other characterizations of the Fisher price index can be found in Funke and Voeller (1978) and Balk (1985) (1995).

[72] However, we shall show later in section H.5 and Appendix 3.3 that the Törnqvist index approximates the Fisher index quite closely using "normal" time series data that is subject to relatively smooth trends. Hence under these circumstances, the Törnqvist index can be regarded as passing the 20 tests to a reasonably high degree of approximation.

[73] This assertion needs to be qualified: there are many other tests which we have not discussed and price statisticians could differ on the importance of satisfying various sets of tests. Some references which discuss other tests are Eichhorn and Voeller (1976), Balk (1995) and Vogt and Barta (1997).

90. There is an additional test that many national income accountants regard as very important: the *additivity test*. This is a test or property that is placed on the implicit quantity index $Q(p^0,p^1,q^0,q^1)$ that corresponds to the price index $P(p^0,p^1,q^0,q^1)$ using the product test (3.3). This test states that the implicit quantity index has the following form:

(3.49) $Q(p^0,p^1,q^0,q^1) = \sum_{i=1}^{n} p_i{*}q_i^1 / \sum_{m=1}^{n} p_m{*}q_m^0$

where the common across periods *price* for commodity i, $p_i{*}$ for $i = 1,\ldots,n$, can be a function of all 4n prices and quantities pertaining to the two periods or situations under consideration, p^0,p^1,q^0,q^1. In the literature on making multilateral comparisons (i.e., comparisons between more than two situations), it is quite common to assume that the quantity comparison between any two regions can be made using the two regional quantity vectors, q^0 and q^1, and a common reference price vector, $p^* \equiv (p_1{*},\ldots,p_n{*})$.[74]

91. Obviously, different versions of the additivity test can be obtained if we place further restrictions on precisely which variables each reference price $p_i{*}$ depends. The simplest such restriction is to assume that each $p_i{*}$ depends only on the commodity i prices pertaining to each of the two situations under consideration, p_i^0 and p_i^1. If we further assume that the functional form for the weighting function is the same for each commodity, so that $p_i{*} = m(p_i^0,p_i^1)$ for $i = 1,\ldots,n$, then we are led to the *unequivocal quantity index* postulated by Knibbs (1924; 44).

92. The theory of the *unequivocal quantity index* (or the *pure quantity index*[75]) parallels the theory of the pure price index outlined in section C.2 above. We give a brief outline of this theory. Let the pure quantity index Q_K have the following functional form:

(3.50) $Q_K(p^0,p^1,q^0,q^1) \equiv \sum_{i=1}^{n} q_i^1\, m(p_i^0,p_i^1) / \sum_{k=1}^{n} q_k^0\, m(p_k^0,p_k^1)$.

We assume that the price vectors p^0 and p^1 are strictly positive and the quantity vectors q^0 and q^1 are nonnegative but have at least one positive component.[76] Our problem is to determine the functional form for the averaging function m if possible. To do this, we need to impose some tests or properties on the pure quantity index Q_K. As was the case with the pure price index, it is very reasonable to ask that the quantity index satisfy the *time reversal test*:

(3.51) $Q_K(p^1,p^0,q^1,q^0) = 1/Q_K(p^0,p^1,q^0,q^1)$.

As was the case with the theory of the unequivocal price index, it can be seen that if the unequivocal quantity index Q_K is to satisfy the time reversal test (3.51), the mean

[74] Hill (1993; 395-397) termed such multilateral methods *the block approach* while Diewert (1996; 250-251) (1999) used the term *average price approaches*. Diewert (1999; 19) used the term *additive multilateral system*. For axiomatic approaches to multilateral index number theory, see Balk (1996a) and Diewert (1999).

[75] Diewert (2001) used this term.

[76] We assume that m(a,b) has the following two properties: m(a,b) is a positive and continuous function, defined for all positive numbers a and b and m(a,a) = a for all a > 0.

function in (3.50) must be *symmetric*. We also ask that Q_K satisfy the following *invariance to proportional changes in current prices test*.

$$(3.52)\quad Q_K(p^0,\lambda p^1,q^0,q^1) = Q_K(p^0,p^1,q^0,q^1) \text{ for all } p^0,p^1,q^0,q^1 \text{ and all } \lambda > 0.$$

The idea behind the invariance test (3.52) is this: the quantity index $Q_K(p^0,p^1,q^0,q^1)$ should only depend on the *relative* prices in each period and it should not depend on the amount of inflation between the two periods. Another way to interpret test (3.52) is to look at what the test implies for the corresponding implicit price index, P_{IK}, defined using the product test (3.5). It can be shown that if Q_K satisfies (3.52), then the corresponding implicit price index P_{IK} will satisfy test T5 above, the *proportionality in current prices test*. The two tests, (3.51) and (3.52), enable us to determine the precise functional form for the pure quantity index Q_K defined by (3.50) above: the *pure quantity index* or Knibbs' *unequivocal quantity index* Q_K must be the Walsh quantity index Q_W[77] defined by:

$$(3.53)\quad Q_W(p^0,p^1,q^0,q^1) \equiv \sum_{i=1}^n q_i^1(p_i^0 p_i^1)^{1/2} / \sum_{k=1}^n q_k^0(p_k^0 p_k^1)^{1/2} .$$

93. Thus with the addition of two tests, the pure price index P_K must be the Walsh price index P_W defined by (3.19) and with the addition of the same two tests (but applied to quantity indices instead of price indices), the pure quantity index Q_K must be the Walsh quantity index Q_W defined by (3.53). However, note that the product of the Walsh price and quantity indices is *not* equal to the expenditure ratio, V^1/V^0. Thus believers in the pure or unequivocal price and quantity index concepts *have to choose one of these two concepts*; they both cannot apply simultaneously.[78]

94. If the quantity index $Q(p^0,p^1,q^0,q^1)$ satisfies the additivity test (3.49) for some price weights p_i^*, then we can rewrite the percentage change in the quantity aggregate, $Q(p^0,p^1,q^0,q^1) - 1$, as follows:

$$\begin{aligned}(3.54)\quad Q(p^0,p^1,q^0,q^1) - 1 &= \{\textstyle\sum_{i=1}^n p_i^*q_i^1 / \sum_{m=1}^n p_m^*q_m^0\} - 1\\ &= \{\textstyle\sum_{i=1}^n p_i^*q_i^1 - \sum_{m=1}^n p_i^*q_i^0\} / \sum_{m=1}^n p_m^*q_m^0\\ &= \textstyle\sum_{i=1}^n w_i\{q_i^1 - q_i^0\}\end{aligned}$$

where the *weight* for commodity i, w_i, is defined as

$$(3.55)\quad w_i \equiv p_i^* / \sum_{m=1}^n p_m^*q_m^0 \quad ; i = 1,\ldots,n.$$

Note that the change in commodity i going from situation 0 to situation 1 is $q_i^1 - q_i^0$. Thus the ith term on the right hand side of (3.54) is the *contribution of the change in commodity i to the overall percentage change in the aggregate going from period 0 to 1*. Business analysts often want statistical agencies to provide decompositions like (3.54) above so that they can decompose the overall change in an aggregate into sector specific

[77] This is the quantity index that corresponds to the price index 8 defined by Walsh (1921; 101).
[78] Knibbs (1924) did not notice this point!

components of change.[79] Thus there is a demand on the part of users for additive quantity indices.

95. For the Walsh quantity index defined by (3.53), the ith weight is

$$(3.56)\quad w_{Wi} \equiv [p_i^0 p_i^1]^{1/2} / \sum_{m=1}^n [p_m^0 p_m^1]^{1/2} q_m^0 \quad ; i = 1,\ldots,n.$$

Thus the Walsh quantity index Q_W has a percentage decomposition into component changes of the form (3.54) where the weights are defined by (3.56).

96. It turns out that the Fisher quantity index Q_F defined by (3.14) also has an additive percentage change decomposition of the form given by (3.54).[80] The ith weight w_{Fi} for this Fisher decomposition is rather complicated and depends on the Fisher quantity index $Q_F(p^0,p^1,q^0,q^1)$ as follows[81]:

$$(3.57)\quad w_{Fi} \equiv [w_i^0 + \{Q_F\}^2 w_i^1]/[1 + Q_F] \; ; \quad i = 1,\ldots,n$$

where Q_F is the value of the Fisher quantity index, $Q_F(p^0,p^1,q^0,q^1)$ and the period t normalized price for commodity i, w_i^t, is defined as the period i price p_i^t divided by the period t expenditure on the aggregate:

$$(3.58)\quad w_i^t \equiv p_i^t / \sum_{m=1}^n p_m^t q_m^t \quad ; \; t = 0,1 \; ; \quad i = 1,\ldots,n.$$

Using the weights w_{Fi} defined by (3.57) and (3.58), we obtain the following exact decomposition for the Fisher ideal quantity index[82]:

$$(3.59)\quad Q_F(p^0,p^1,q^0,q^1) - 1 = \sum_{i=1}^n w_{Fi}\{q_i^1 - q_i^0\}.$$

Thus the lack of additivity of the Fisher quantity index does not prevent it from having an additive percentage change decomposition.

97. Due to the symmetric nature of the Fisher price and quantity indices, it can be seen that the Fisher price index P_F defined by (3.12) also has the following additive percentage change decomposition:

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[79] Business and government analysts also often demand an analogous decomposition of the change in price aggregate into sector specific components that add up.

[80] The Fisher quantity index also has an additive decomposition of the type defined by (3.49) due to Van Ijzeren (1987; 6). The ith reference price p_i^* is defined as $p_i^* \equiv (1/2)p_i^0 + (1/2)p_i^1/P_F(p^0,p^1,q^0,q^1)$ for i = 1,…,n and where P_F is the Fisher price index. This decomposition was also independently derived by Dikhanov (1997). The Van Ijzeren decomposition for the Fisher quantity index is currently being used by Bureau of Economic Analysis; see Moulton and Seskin (1999; 16) and Ehemann, Katz and Moulton (2000).

[81] This decomposition was obtained by Diewert (2000b) and Reinsdorf, Diewert and Ehemann (2001). For an economic interpretation of this decomposition, see Diewert (2000b).

[82] To verify the exactness of the decomposition, substitute (3.57) into (3.59) and solve the resulting equation for Q_F. We find that the solution is equal to Q_F defined by (3.14) above.

(3.60) $P_F(p^0,p^1,q^0,q^1) - 1 = \sum_{i=1}^n v_{Fi}\{p_i^1 - p_i^0\}$

where the commodity i weight v_{Fi} is defined as

(3.61) $v_{Fi} \equiv [v_i^0 + \{P_F\}^2 v_i^1]/[1 + P_F]$; $i = 1,\ldots,n$

where P_F is the value of the Fisher price index, $P_F(p^0,p^1,q^0,q^1)$ and the period t normalized quantity for commodity i, v_i^t, is defined as the period i quantity q_i^t divided by the period t expenditure on the aggregate:

(3.62) $v_i^t \equiv q_i^t / \sum_{m=1}^n p_m^t q_m^t$; $t = 0,1$; $i = 1,\ldots,n$.

The above results show that while the Fisher price and quantity indices do *not* satisfy the additivity test, the percentage change in each of these indices *does* have an exact additive decomposition into components that give the contribution to the overall change in the price (or quantity) index of the change in each price (or quantity).

G. The stochastic approach to price indices

G.1 The early unweighted stochastic approach

98. The stochastic approach to the determination of the price index can be traced back to the work of Jevons and Edgeworth over a hundred years ago[83]. The basic idea behind the (unweighted) stochastic approach is that each price relative, p_i^1/p_i^0 for $i = 1,2,\ldots,n$ can be regarded as an estimate of a common inflation rate α between periods 0 and 1[84]; i.e., it is assumed that

(3.63) $p_i^1/p_i^0 = \alpha + \varepsilon_i$; $i = 1,2,\ldots,n$

where α is the common inflation rate and the ε_i are random variables with mean 0 and variance σ^2. The least squares or maximum likelihood estimator for α is the Carli (1764) price index P_C defined as

(3.64) $P_C(p^0,p^1) \equiv \sum_{i=1}^n (1/n)\, p_i^1/p_i^0$.

A drawback of the Carli price index is that it does not satisfy the time reversal test, i.e., $P_C(p^1,p^0) \neq 1/ P_C(p^0,p^1)$[85].

[83] For references to the literature, see Diewert (1993a; 37-38) (1995a) (1995b).
[84] "In drawing our averages the independent fluctuations will more or less destroy each other; the one required variation of gold will remain undiminished." W. Stanley Jevons (1884; 26).
[85] In fact Fisher (1922; 66) noted that $P_C(p^0,p^1)\, P_C(p^1,p^0) \geq 1$ unless the period 1 price vector p^1 is proportional to the period 0 price vector p^0; i.e., Fisher showed that the Carli index has a definite upward bias. He urged statistical agencies not to use this formula.

99. Let us change our stochastic specification and assume that the logarithm of each price relative, $\ln(p_i^1/p_i^0)$, is an unbiased estimate of the logarithm of the inflation rate between periods 0 and 1, β say. The counterpart to (3.63) is now:

(3.65) $\ln(p_i^1/p_i^0) = \beta + \varepsilon_i$; $i = 1,2,\ldots,n$

where $\beta \equiv \ln\alpha$ and the ε_i are independently distributed random variables with mean 0 and variance σ^2. The least squares or maximum likelihood estimator for β is the logarithm of the geometric mean of the price relatives. Hence the corresponding estimate for the common inflation rate α[86] is the Jevons (1865) price index P_J defined earlier by (3.42); namely as: $P_J(p^0,p^1) \equiv \prod_{i=1}^n (p_i^1/p_i^0)^{1/n}$.

100. The Jevons price index P_J does satisfy the time reversal test and hence is much more satisfactory than the Carli index P_C. However, both the Jevons and Carli price indices suffer from a fatal flaw: each price relative p_i^1/p_i^0 is regarded as *being equally important* and is given an equal weight in the index number formulae (3.64) and (3.65). Keynes was particularly critical of this *unweighted stochastic approach* to index number theory. He directed the following criticism towards this approach, which was vigorously advocated by Edgeworth (1923):

"Nevertheless I venture to maintain that such ideas, which I have endeavoured to expound above as fairly and as plausibly as I can, are root-and-branch erroneous. The 'errors of observation', the 'faulty shots aimed at a single bull's eye' conception of the index number of prices, Edgeworth's 'objective mean variation of general prices', is the result of confusion of thought. There is no bull's eye. There is no moving but unique centre, to be called the general price level or the objective mean variation of general prices, round which are scattered the moving price levels of individual things. There are all the various, quite definite, conceptions of price levels of composite commodities appropriate for various purposes and inquiries which have been scheduled above, and many others too. There is nothing else. Jevons was pursuing a mirage.

What is the flaw in the argument? In the first place it assumed that the fluctuations of individual prices round the 'mean' are 'random' in the sense required by the theory of the combination of independent observations. In this theory the divergence of one 'observation' from the true position is assumed to have no influence on the divergences of other 'observations'. But in the case of prices, a movement in the price of one commodity necessarily influences the movement in the prices of other commodities, whilst the magnitudes of these compensatory movements depend on the magnitude of the change in expenditure on the first commodity as compared with the importance of the expenditure on the commodities secondarily affected. Thus, instead of 'independence', there is between the 'errors' in the successive 'observations' what some writers on probability have called 'connexity', or, as Lexis expressed it, there is 'sub-normal dispersion'.

We cannot, therefore, proceed further until we have enunciated the appropriate law of connexity. But the law of connexity cannot be enunciated without reference to the relative importance of the commodities

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[86] Greenlees (1999) pointed out that although $(1/n) \sum_{i=1}^n \ln(p_i^1/p_i^0)$ is an unbiased estimator for β, the corresponding exponential of this estimator, P_J defined by (3.65), will generally *not* be an unbiased estimator for α under our stochastic assumptions. To see this, let $x_i = \ln p_i^1/p_i^0$. Taking expectations, we have: $E\, x_i = \beta = \ln \alpha$. Define the positive, convex function f of one variable x by $f(x) \equiv e^x$. By Jensen's (1906) inequality, we have $Ef(x) \geq f(Ex)$. Letting x equal the random variable x_i, this inequality becomes: $E(p_i^1/p_i^0) = Ef(x_i) \geq f(E\, x_i) = f(\beta) = e^\beta = e^{\ln\alpha} = \alpha$. Thus for each n, we have $E(p_i^1/p_i^0) \geq \alpha$, and it can be seen that the Jevons price index defined by (3.65) will generally have an upward bias under the usual stochastic assumptions.

affected—which brings us back to the problem that we have been trying to avoid, of weighting the items of a composite commodity." John Maynard Keynes (1930; 76-77).

The main point Keynes seemed to be making in the above quotation is that prices in the economy are not independently distributed from each other and from quantities. In current macroeconomic terminology, we can interpret Keynes as saying that a macroeconomic shock will be distributed across all prices and quantities in the economy through the normal interaction between supply and demand; i.e., through the workings of the general equilibrium system. Thus Keynes seemed to be leaning towards the economic approach to index number theory (even before it was even developed to any great extent), where quantity movements are functionally related to price movements. A second point that Keynes made in the above quotation is that there is no such thing as *the* inflation rate; there are only price changes that pertain to well specified sets of commodities or transactions; i.e., the domain of definition of the price index must be carefully specified.[87] A final point that Keynes made is that price movements must be weighted by their economic importance; i.e., by quantities or expenditures.

101. In addition to the above theoretical criticisms, Keynes also made the following strong empirical attack on Edgeworth's unweighted stochastic approach:

"The Jevons—Edgeworth "objective mean variation of general prices', or 'indefinite' standard, has generally been identified, by those who were not as alive as Edgeworth himself was to the subtleties of the case, with the purchasing power of money—if only for the excellent reason that it was difficult to visualise it as anything else. And since any respectable index number, however weighted, which covered a fairly large number of commodities could, in accordance with the argument, be regarded as a fair approximation to the indefinite standard, it seemed natural to regard any such index as a fair approximation to the purchasing power of money also.

Finally, the conclusion that all the standards 'come to much the same thing in the end' has been reinforced 'inductively' by the fact that rival index numbers (all of them, however, of the wholesale type) have shown a considerable measure of agreement with one another in spite of their different compositions. … On the contrary, the tables given above (pp. 53,55) supply strong presumptive evidence that over long period as well as over short period the movements of the wholesale and of the consumption standards respectively are capable of being widely divergent." John Maynard Keynes (1930; 80-81).

In the above quotation, Keynes noted that the proponents of the unweighted stochastic approach to price change measurement were comforted by the fact that all of the then existing (unweighted) indices of wholesale prices showed broadly similar movements. However, Keynes showed empirically that his wholesale price indices moved quite differently than his consumer price indices.

102. In order to overcome the Keynesian criticisms of the unweighted stochastic approach to index numbers, it is necessary to:

- have a definite domain of definition for the index number and
- weight the price relatives by their economic importance.[88]

[87] See section B.1 above.

[88] Walsh (1901) (1921; 82-83) also objected to the lack of weighting in the unweighted stochastic approach to index number theory.

We turn now to a discussion of alternative methods of weighting.

G.2 The weighted stochastic approach

103. Walsh seems to have been the first index number theorist to point out that a sensible stochastic approach to measuring price change means that individual price relatives should be weighted according to their economic importance or *their transactions value* in the two periods under consideration:

"It might seem at first sight as if simply every price quotation were a single item, and since every commodity (any kind of commodity) has one price-quotation attached to it, it would seem as if price-variations of every kind of commodity were the single item in question. This is the way the question struck the first inquirers into price-variations, wherefore they used simple averaging with even weighting. But a price-quotation is the quotation of the price of a generic name for many articles; and one such generic name covers a few articles, and another covers many. … A single price-quotation, therefore, may be the quotation of the price of a hundred, a thousand, or a million dollar's worths, of the articles that make up the commodity named. Its weight in the averaging, therefore, ought to be according to these money-unit's worth." Correa Moylan Walsh (1921; 82-83).

However, Walsh did not specify exactly how these economic weights should be determined.

104. Theil (1967; 136-137) proposed a solution to the lack of weighting in the Jevons index, (3.65). He argued as follows. Suppose we draw price relatives at random in such a way that each dollar of expenditure in the base period has an equal chance of being selected. Then the probability that we will draw the ith price relative is equal to $s_i^0 \equiv p_i^0 q_i^0 / \sum_{k=1}^n p_k^0 q_k^0$, the period 0 expenditure share for commodity i. Then the overall mean (period 0 weighted) logarithmic price change is $\sum_{i=1}^n s_i^0 \ln(p_i^1/p_i^0)$.[89] Now repeat the above mental experiment and draw price relatives at random in such a way that each dollar of expenditure in period 1 has an equal probability of being selected. This leads to the overall mean (period 1 weighted) logarithmic price change of $\sum_{i=1}^n s_i^1 \ln(p_i^1/p_i^0)$.[90] Each of these measures of overall logarithmic price change seems equally valid so we could argue for taking a symmetric average of the two measures in order to obtain a final single measure of overall logarithmic price change. Theil[91] argued that a nice symmetric index number formula can be obtained if we make the probability of selection for the nth price relative equal to the arithmetic average of the period 0 and 1 expenditure shares for commodity n. Using these probabilities of selection, Theil's final measure of overall logarithmic price change was

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[89] In Appendix 3.3, we will follow the terminology introduced by Vartia (1978; 272) and refer to this index as the *logarithmic Laspeyres index*, P_{LL}. This terminology is briefer than *the base weighted geometric index*. An alternative terminology might be *the geometric Laspeyres index*.

[90] In Appendix 3.3, we will follow the terminology introduced by Vartia (1978; 272) and refer to this index as the *logarithmic Paasche index*, P_{LP}.

[91] "The price index number defined in (1.8) and (1.9) uses the n individual logarithmic price differences as the basic ingredients. They are combined linearly by means of a two stage random selection procedure: First, we give each region the same chance _ of being selected, and second, we give each dollar spent in the selected region the same chance ($1/m_a$ or $1/m_b$) of being drawn." Henri Theil (1967; 138).

(3.66) $\ln P_T(p^0,p^1,q^0,q^1) \equiv \sum_{i=1}^n (1/2)(s_i^0 + s_i^1) \ln(p_i^1/p_i^0)$.

Note that the index P_T defined by (3.66) is equal to the Törnqvist index defined earlier by (3.43).

105. We can give the following statistical interpretation of the right hand side of (3.66). Define the ith logarithmic price ratio r_i by:

(3.67) $r_i \equiv \ln(p_i^1/p_i^0)$ for $i = 1,\ldots,n$.

Now define the discrete random variable, R say, as the random variable which can take on the values r_i with probabilities $\rho_i \equiv (1/2)[s_i^0 + s_i^1]$ for $i = 1,\ldots,n$. Note that since each set of expenditure shares, s_i^0 and s_i^1, sums to one over i, the probabilities ρ_i will also sum to one. It can be seen that the expected value of the discrete random variable R is

(3.68) $E[R] \equiv \sum_{i=1}^n \rho_i r_i = \sum_{i=1}^n (1/2)(s_i^0 + s_i^1) \ln(p_i^1/p_i^0) = \ln P_T(p^0,p^1,q^0,q^1)$

using (3.66) and (3.65). Thus the logarithm of the index P_T can be interpreted as *the expected value of the distribution of the logarithmic price ratios* in the domain of definition under consideration, where the n discrete price ratios in this domain of definition are weighted according to Theil's probability weights, $\rho_i \equiv (1/2)[s_i^0 + s_i^1]$ for $i = 1,\ldots,n$.

106. Taking antilogs of both sides of (3.66), we obtain the Törnqvist (1936) (1937) Theil price index, P_T.[92] This index number formula has a number of good properties. In particular, P_T satisfies the proportionality in current prices test T5 and the time reversal test T11 discussed in section F above. These two tests can be used to justify Theil's (arithmetic) method of forming an average of the two sets of expenditure shares in order to obtain his probability weights, $\rho_i \equiv (1/2)[s_i^0 + s_i^1]$ for $i = 1,\ldots,n$. Consider the following *symmetric mean class of logarithmic index number formulae*:

(3.69) $\ln P_s(p^0,p^1,q^0,q^1) \equiv \sum_{i=1}^n m(s_i^0,s_i^1) \ln(p_i^1/p_i^0)$

where $m(s_i^0,s_i^1)$ is a positive function of the period 0 and 1 expenditure shares on commodity i, s_i^0 and s_i^1 respectively. In order for P_s to satisfy the time reversal test, it is necessary that the function m be symmetric. Then it can be shown[93] that for P_s to satisfy test T5, m must be the arithmetic mean. This provides a reasonably strong justification for Theil's choice of the mean function.

[92] The sampling bias problem studied by Greenlees (1999) does not occur in the present context because there is no sampling involved in definition (3.66): the sum of the $p_i^t q_i^t$ over i for each period t is assumed to equal the value aggregate V^t for period t.

[93] See Diewert (2000a) and Balk and Diewert (2001).

107. The stochastic approach of Theil has another nice symmetry property. Instead of considering the distribution of the price ratios $r_i = \ln p_i^1/p_i^0$, we could also consider the distribution of the *reciprocals* of these price ratios, say:

$$(3.70)\quad \begin{aligned} t_i &\equiv \ln p_i^0/p_i^1 \quad \text{for } i = 1,\ldots,n \\ &= \ln (p_i^1/p_i^0)^{-1} \\ &= -\ln (p_i^1/p_i^0) \\ &= -r_i \end{aligned}$$

where the last equality follows using definitions (3.67). We can still associate the symmetric probability, $\rho_i \equiv (1/2)[s_i^0 + s_i^1]$, with the ith reciprocal logarithmic price ratio t_i for i = 1,…,n. Now define the discrete random variable, T say, as the random variable which can take on the values t_i with probabilities $\rho_i \equiv (1/2)[s_i^0 + s_i^1]$ for i = 1,…,n. It can be seen that the expected value of the discrete random variable T is

$$(3.71)\quad \begin{aligned} E[T] &\equiv \textstyle\sum_{i=1}^n \rho_i t_i \\ &= -\textstyle\sum_{i=1}^n \rho_i r_i && \text{using (3.70)} \\ &= -E[R] && \text{using (3.68)} \\ &= -\ln P_T(p^0,p^1,q^0,q^1). \end{aligned}$$

Thus it can be seen that the distribution of the random variable T is equal to minus the distribution of the random variable R. Hence it does not matter whether we consider the distribution of the original logarithmic price ratios, $r_i \equiv \ln p_i^1/p_i^0$, or the distribution of their reciprocals, $t_i \equiv \ln p_i^0/p_i^1$: we obtain essentially the same stochastic theory.

108. It is possible to consider weighted stochastic approaches to index number theory where we look at the distribution of the price ratios, p_i^1/p_i^0, rather than the distribution of the logarithmic price ratios, $\ln p_i^1/p_i^0$. Thus, again following in the footsteps of Theil, suppose we draw price relatives at random in such a way that each dollar of expenditure in the *base period* has an equal chance of being selected. Then the probability that we will draw the ith price relative is equal to s_i^0, the period 0 expenditure share for commodity i. Now the overall mean (period 0 weighted) price change is:

$$(3.72)\quad P_L(p^0,p^1,q^0,q^1) = \textstyle\sum_{i=1}^n s_i^0(p_i^1/p_i^0),$$

which turns out to be the Laspeyres price index, P_L (recall (3.8) above). This stochastic approach is the natural one for studying *sampling problems* associated with implementing a Laspeyres price index.

109. Now repeat the above mental experiment and draw price relatives at random in such a way that each dollar of expenditure in period 1 has an equal probability of being selected. This leads to the overall mean (period 1 weighted) price change equal to:

$$(3.73)\quad P_{Pal}(p^0,p^1,q^0,q^1) = \textstyle\sum_{i=1}^n s_i^1(p_i^1/p_i^0).$$

This is known as the Palgrave (1886) index number formula.[94]

110. It can be verified that neither the Laspeyres nor Palgrave price indices satisfy the time reversal test, T11. Thus, again following in the footsteps of Theil, we might try to obtain a formula that satisfied the time reversal test by taking a symmetric average of the two sets of shares. Thus we consider the following class of *symmetric mean index number formulae*:

(3.74) $P_m(p^0,p^1,q^0,q^1) \equiv \sum_{i=1}^n m(s_i^0,s_i^1)\,(p_i^1/p_i^0)$

where $m(s_i^0,s_i^1)$ is a symmetric function of the period 0 and 1 expenditure shares for commodity i, s_i^0 and s_i^1 respectively. In order to interpret the right hand side of (3.74) as an expected value of the price ratios p_i^1/p_i^0, it is necessary that

(3.75) $\sum_{i=1}^n m(s_i^0,s_i^1) = 1.$

However, in order to satisfy (3.75), m must be the arithmetic mean.[95] With this choice of m, (3.75) becomes the following (unnamed) index number formula, P_u:

(3.76) $P_u(p^0,p^1,q^0,q^1) \equiv \sum_{i=1}^n (1/2)[s_i^0 + s_i^1](p_i^1/p_i^0).$

Unfortunately, the unnamed index P_u does not satisfy the time reversal test either.[96]

111. Instead of considering the distribution of the price ratios, p_i^1/p_i^0, we could also consider the distribution of the *reciprocals* of these price ratios. The counterparts to the asymmetric indices defined earlier by (3.72) and (3.73) are now $\sum_{i=1}^n s_i^0(p_i^0/p_i^1)$ and $\sum_{i=1}^n s_i^1(p_i^0/p_i^1)$ respectively. These are (stochastic) price indices going *backwards* from period 1 to 0. In order to make these indices comparable with our previous forward looking indices, we take the reciprocals of these indices (which leads to harmonic averages) and obtain the following two indices:

(3.77) $P_{HL}(p^0,p^1,q^0,q^1) \equiv [\sum_{i=1}^n s_i^0(p_i^0/p_i^1)]^{-1}$;

(3.78) $$\begin{aligned} P_{HP}(p^0,p^1,q^0,q^1) &\equiv [\sum_{i=1}^n s_i^1(p_i^0/p_i^1)]^{-1} \\ &= [\sum_{i=1}^n s_i^1(p_i^1/p_i^0)^{-1}]^{-1} \\ &= P_P(p^0,p^1,q^0,q^1) \qquad \text{using (3.9) above.} \end{aligned}$$

Thus the reciprocal stochastic price index defined by (3.78) turns out to equal the fixed basket Paasche price index, P_P, defined earlier by (3.9). This stochastic approach is the natural one for studying *sampling problems* associated with implementing a Paasche

[94] It is formula number 9 in Fisher's (1922; 466) listing of index number formulae.

[95] For a proof of this assertion, see Balk and Diewert (2001).

[96] In fact, this index suffers from the same upward bias as the Carli index in that we have $P_u(p^0,p^1,q^0,q^1)P_u(p^1,p^0,q^1,q^0) \geq 1$. To prove this, note that the previous inequality is equivalent to $[P_u(p^1,p^0,q^1,q^0)]^{-1} \leq P_u(p^0,p^1,q^0,q^1)$ and this inequality follows from the fact that a weighted harmonic mean of n positive numbers is equal or less than the corresponding weighted arithmetic mean.

price index. The other asymmetrically weighted reciprocal stochastic price index defined by (3.77) has no author's name associated with it but it was noted by Irving Fisher (1922; 467) as his index number formula 13. Vartia (1978;272) called this index *the harmonic Laspeyres index* and we will use his terminology. We can also consider the class of *symmetrically weighted reciprocal price indices* defined as:

(3.79) $P_{mr}(p^0,p^1,q^0,q^1) \equiv [\sum_{i=1}^{n} m(s_i^0,s_i^1)\,(p_i^1/p_i^0)^{-1}\,]^{-1}$

where as usual, $m(s_i^0,s_i^1)$ is a homogeneous symmetric mean of the period 0 and 1 expenditure shares on commodity i. However, none of the indices defined by (3.77) – (3.79) satisfy the time reversal test.

112. The fact that Theil's index number formula P_T satisfies the time reversal test leads us to prefer Theil's index as the "best" weighted stochastic approach.

113. The main features of the weighted stochastic approach to index number theory can be summarized as follows. It is first necessary to pick two periods and a transactions domain of definition. As usual, each value transaction for each of the n commodities in our domain of definition is split up into price and quantity components. Then, assuming there are no new commodities or no disappearing commodities, we have n price relatives p_i^1/p_i^0 pertaining to the two situations under consideration along with the corresponding 2n expenditure shares. The weighted stochastic approach just assumes that these n relative prices, or some transformation of these price relatives $f(p_i^1/p_i^0)$, have a discrete statistical distribution, where the ith probability, $\rho_i = m(s_i^0,s_i^1)$, is a function of the expenditure shares pertaining to commodity i in the two situations under consideration, s_i^0 and s_i^1. Different price indices result, depending on how one chooses the functions f and m. In Theil's approach, the transformation function f was the natural logarithm and the mean function m was the simple unweighted arithmetic mean.

114. There is a third aspect to the weighted stochastic approach to index number theory and that is we have to decide what *single number* best summarizes the distribution of the n (possibly transformed) price relatives. We chose the *mean* of the discrete distribution as our "best" summary measure for the distribution of the (possibly transformed) price relatives but other measures are possible. In particular, the *weighted median* or various *trimmed means* are often suggested as the "best" measure of central tendency because these measures minimize the influence of outliers. However, a detailed discussion of these alternative measures of central tendency is beyond the scope of this chapter. Additional material on stochastic approaches to index number theory and references to the literature can be found in Clements and Izan (1981) (1987), Selvanathan and Rao (1994), Diewert (1995b), Cecchetti (1997) and Wynne (1997) (1999).

H. Economic Approaches: The case of one household

H.1 The Konüs cost of living index and observable bounds

115. In this subsection, we will outline the theory of the cost of living index for a single consumer (or household) that was first developed by the Russian economist, A. A. Konüs (1924). This theory relies on the assumption of *optimizing behavior* on the part of economic agents (consumers or producers). Thus given a vector of commodity or input prices p^t that the agent faces in a given time period t, it is assumed that the corresponding observed quantity vector q^t is the solution to a cost minimization problem that involves either the consumer's preference or utility function f or the producer's production function f.[97] Thus in contrast to the axiomatic approach to index number theory, the economic approach does *not* assume that the two quantity vectors q^0 and q^1 are independent of the two price vectors p^0 and p^1. In the economic approach, the period 0 quantity vector q^0 is determined by the consumer's preference function f and the period 0 vector of prices p^0 that the consumer faces and the period 1 quantity vector q^1 is determined by the consumer's preference function f and the period 1 vector of prices p^1.

116. We assume that "the" consumer has well defined *preferences* over different combinations of the n consumer commodities or items.[98] Each combination of items can be represented by a positive vector $q \equiv [q_1,\ldots,q_n]$. The consumer's preferences over alternative possible consumption vectors q are assumed to be representable by a continuous, nondecreasing and concave[99] utility function f. Thus if $f(q^1) > f(q^0)$, then the consumer prefers the consumption vector q^1 to q^0. We further assume that the consumer minimizes the cost of achieving the period t utility level $u^t \equiv f(q^t)$ for periods t = 0,1. Thus we assume that the observed period t consumption vector q^t solves the following *period t cost minimization problem*:

$$(3.80)\quad C(u^t,p^t) \equiv \min_q \{\textstyle\sum_{i=1}^n p_i^t q_i : f(q) = u^t \equiv f(q^t)\} = \textstyle\sum_{i=1}^n p_i^t q_i^t ; \quad t = 0,1.$$

The period t price vector for the n commodities under consideration that the consumer faces is p^t. Note that the solution to the cost or expenditure minimization problem (3.80) for a general utility level u and general vector of commodity prices p defines the *consumer's cost function*, C(u,p). We shall use the cost function in order to define the consumer's cost of living price index.

117. The Konüs (1924) family of *true cost of living indices* pertaining to two periods where the consumer faces the strictly positive price vectors $p^0 \equiv (p_1^0,\ldots,p_n^0)$ and $p^1 \equiv (p_1^1,\ldots,p_n^1)$ in periods 0 and 1 respectively is defined as the ratio of the minimum costs of achieving the same utility level $u \equiv f(q)$ where $q \equiv (q_1,\ldots,q_n)$ is a positive reference quantity vector; i.e., we have

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[97] For a description of the economic theory of the input and output price indices, see Balk (1998). In the economic theory of the output price index, q^t is assumed to be the solution to a revenue maximization problem involving the output price vector p^t.

[98] In section H, these preferences are assumed to be invariant over time. In section I, this assumption will be relaxed (one of the environmental variables could be a time variable that shifts tastes).

[99] f is concave if and only if $f(\lambda q^1 + (1-\lambda)q^2) \geq \lambda f(q^1) + (1-\lambda)f(q^2)$ for all $0 \leq \lambda \leq 1$ and all $q^1 >> 0_n$ and $q^2 >> 0_n$. Note that $q \geq 0_N$ means that each component of the N dimensional vector q is nonnegative, $q >> 0_N$ means that each component of q is positive and $q > 0_N$ means that $q \geq 0_N$ but $q \neq 0_N$; i.e., q is nonnegative but at least one component is positive.

(3.81) $P_K(p^0,p^1,q) \equiv C[f(q),p^1] / C[f(q),p^0]$.

We say that definition (3.81) defines a *family* of price indices because there is one such index for each reference quantity vector q chosen.

118. It is natural to choose two specific reference quantity vectors q in definition (3.81): the observed base period quantity vector q^0 and the current period quantity vector q^1. The first of these two choices leads to the *following Laspeyres-Konüs true cost of living index*:

(3.82) $P_K(p^0,p^1,q^0) \equiv C[f(q^0),p^1] / C[f(q^0),p^0]$
$= C[f(q^0),p^1] / \sum_{i=1}^n p_i^0 q_i^0$ using (3.80) for t = 0
$= \min_q \{\sum_{i=1}^n p_i^1 q_i : f(q) = f(q^0)\} / \sum_{i=1}^n p_i^0 q_i^0$
using the definition of the cost minimization problem that defines $C[f(q^0),p^1]$
$\leq \sum_{i=1}^n p_i^1 q_i^0 / \sum_{i=1}^n p_i^0 q_i^0$
since $q^0 \equiv (q_1^0,\ldots,q_n^0)$ is feasible for the minimization problem
$= P_L(p^0,p^1,q^0,q^1)$

where P_L is the Laspeyres price index defined by (3.5) above. *Thus the (unobservable) Laspeyres-Konüs true cost of living index is bounded from above by the observable Laspeyres price index.*[100]

119. The second of the two natural choices for a reference quantity vector q in definition (3.81) leads to the *following Paasche-Konüs true cost of living index*:

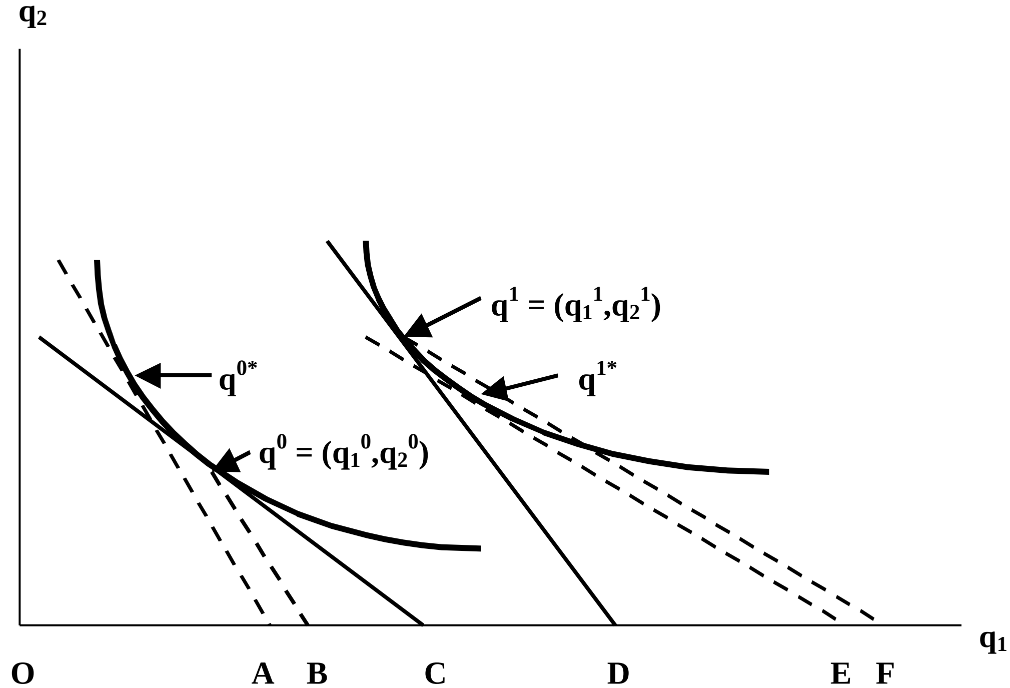
(3.83) $P_K(p^0,p^1,q^1) \equiv C[f(q^1),p^1] / C[f(q^1),p^0]$
$= \sum_{i=1}^n p_i^1 q_i^1 / C[f(q^1),p^0]$ using (3.80) for t = 1
$= \sum_{i=1}^n p_i^1 q_i^1 / \min_q \{\sum_{i=1}^n p_i^0 q_i : f(q) = f(q^1)\}$
using the definition of the cost minimization problem that defines $C[f(q^1),p^0]$
$\geq \sum_{i=1}^n p_i^1 q_i^1 / \sum_{i=1}^n p_i^0 q_i^1$
since $q^1 \equiv (q_1^1,\ldots,q_n^1)$ is feasible for the minimization problem and thus $C[f(q^1),p^0] \leq \sum_{i=1}^n p_i^0 q_i^1$ and hence $1/C[f(q^1),p^0] \geq 1/ \sum_{i=1}^n p_i^0 q_i^1$
$= P_P(p^0,p^1,q^0,q^1)$

where P_P is the Paasche price index defined by (3.6) above. *Thus the (unobservable) Paasche-Konüs true cost of living index is bounded from below by the observable Paasche price index.*[101]

120. It is possible to illustrate the two inequalities (3.82) and (3.83) if there are only two commodities; see Figure 1 below.

[100] This inequality was first obtained by Konüs (1924) (1939; 17). See also Pollak (1983).
[101] This inequality is also due to Konüs (1924) (1939; 19). See also Pollak (1983).

Figure 1: The Laspeyres and Paasche bounds to the true cost of living



The solution to the period 0 cost minimization problem is the vector q^0 and the straight line through C represents the consumer's period 0 budget constraint, the set of quantity points q_1,q_2 such that $p_1^0q_1 + p_2^0q_2 = p_1^0q_1^0 + p_2^0q_2^0$. The curved line through q^0 is the consumer's period 0 indifference curve, the set of points q_1,q_2 such that $f(q_1,q_2) = f(q_1^0,q_2^0)$; i.e., it is the set of consumption vectors that give the same utility as the observed period 0 consumption vector q^0. The solution to the period 1 cost minimization problem is the vector q^1 and the straight line through D represents the consumer's period 1 budget constraint, the set of quantity points q_1,q_2 such that $p_1^1q_1 + p_2^1q_2 = p_1^1q_1^1 + p_2^1q_2^1$. The curved line through q^1 is the consumer's period 1 indifference curve, the set of points q_1,q_2 such that $f(q_1,q_2) = f(q_1^1,q_2^1)$; i.e., it is the set of consumption vectors that give the same utility as the observed period 1 consumption vector q^1. The point q^{0*} solves the hypothetical cost minimization problem of minimizing the cost of achieving the base period utility level $u^0 \equiv f(q^0)$ when facing the period 1 price vector $p^1 = (p_1^1,p_2^1)$. Thus we have $C[u^0,p^1] = p_1^1q_1^{0*} + p_2^1q_2^{0*}$ and the dashed line through A is the corresponding isocost line $p_1^1q_1 + p_2^1q_2 = C[u^0,p^1]$. Note that the hypothetical cost line through A is parallel to the actual period 1 cost line through D. From (3.82), the Laspeyres-Konüs true index is $C[u^0,p^1] / [p_1^0q_1^0 + p_2^0q_2^0]$ while the ordinary Laspeyres index is $[p_1^1q_1^0 + p_2^1q_2^0] / [p_1^0q_1^0 + p_2^0q_2^0]$. Since the denominators for these two indices are the same, the difference between the indices is due to the differences in their numerators. In Figure 1, this difference in the numerators is expressed by the fact that the cost line through A lies *below* the parallel cost line through B. Now if the consumer's indifference curve through the observed period 0 consumption vector q^0 were L shaped with vertex at q^0, then the consumer would not change his or her consumption pattern in response to a change in the relative prices of the two commodities while keeping a fixed standard of living. In this case, the hypothetical vector q^{0*} would coincide with q^0, the dashed line through A would coincide with the dashed line through B and the true Laspeyres-Konüs index would *coincide* with the ordinary Laspeyres index. However, L shaped indifference curves are not generally consistent with consumer behavior; i.e., when the price of a commodity decreases, consumers generally demand more of it. Thus in the general case, there will be a gap between the points A and B. The magnitude of this gap represents the amount of *substitution bias* between the true index and the corresponding Laspeyres index; i.e., the Laspeyres index will generally be *greater* than the corresponding true cost of living index, $P_K(p^0,p^1,q^0)$.

121. Figure 1 can also be used to illustrate the inequality (3.83). First note that the dashed lines through E and F are parallel to the period 0 isocost line through C. The point q^{1*} solves the hypothetical cost minimization problem of minimizing the cost of achieving the current period utility level $u^1 \equiv f(q^1)$ when facing the period 0 price vector $p^0 = (p_1^0,p_2^0)$. Thus we have $C[u^1,p^0] = p_1^0q_1^{1*} + p_2^0q_2^{1*}$ and the dashed line through E is the corresponding isocost line $p_1^1q_1 + p_2^1q_2 = C[u^0,p^1]$. From (3.83), the Paasche-Konüs true index is $[p_1^1q_1^1 + p_2^1q_2^1] / C[u^1,p^0]$ while the ordinary Paasche index is $[p_1^1q_1^1 + p_2^1q_2^1] / [p_1^0q_1^1 + p_2^0q_2^1]$. Since the numerators for these two indices are the same, the difference between the indices is due to the differences in their denominators. In Figure 1, this difference in the denominators is expressed by the fact that the cost line through E lies *below* the parallel cost line through F. The magnitude of this difference represents the amount of *substitution bias* between the true index and the corresponding Paasche index; i.e., the Paasche index will generally be *less* than the corresponding true cost of living index, $P_K(p^0,p^1,q^1)$. Note that this inequality goes in the opposite direction to the previous inequality between the two Laspeyres indices. The reason for this change in direction is due to the fact that one set of differences between the two indices takes place in the numerators of the two indices (the Laspeyres inequalities) while the other set takes place in the denominators of the two indices (the Paasche inequalities).

122. The bound (3.82) on the Laspeyres-Konüs true cost of living $P_K(p^0,p^1,q^0)$ using the base period level of utility as the living standard is *one sided* as is the bound (3.83) on the Paasche-Konüs true cost of living $P_K(p^0,p^1,q^1)$ using the *current period* level of utility as the living standard. In a remarkable result, Konüs (1924; 20) showed that there exists an intermediate consumption vector q* that is on the straight line joining the base period consumption vector q^0 and the current period consumption vector q^1 such that the corresponding (unobservable) true cost of living index $P_K(p^0,p^1,q^*)$ is between the observable Laspeyres and Paasche indices, P_L and P_P.[102] Thus we have the existence of a number λ^* between 0 and 1 such that

$$(3.84)\quad P_L \le P_K(p^0,p^1, \lambda^*q^0 + (1-\lambda^*)q^1) \le P_P \quad \text{or} \quad P_P \le P_K(p^0,p^1, \lambda^*q^0 + (1-\lambda^*)q^1) \le P_L.$$

The above inequalities are of some practical importance. If the observable (in principle) Paasche and Laspeyres indices are not too far apart, then taking a symmetric average of

[102] For more recent applications of the Konüs method of proof, see Diewert (1983a;191) for an application to the consumer context and Diewert (1983b; 1059-1061) for an application to the producer context.

these indices should provide a good approximation to a true cost of living index where the reference standard of living is somewhere between the base and current period living standards. To determine the precise symmetric average of the Paasche and Laspeyres indices, we can appeal to the results in section C.1 above and take the geometric mean, which is the Fisher price index. Thus the Fisher ideal price index receives a fairly strong justification as a good approximation to an unobservable theoretical cost of living index.

123. The bounds (3.82)-(3.84) are the best bounds that we can obtain on true cost of living indices without making further assumptions. In subsequent subsections, we make further assumptions on the class of utility functions that describe the consumer's tastes for the n commodities under consideration. With these extra assumptions, we are able to determine the consumer's true cost of living exactly.

H.2 The true cost of living index when preferences are homothetic

124. Up to now, the consumer's preference function f did not have to satisfy any particular homogeneity assumption. For the remainder of this section, we assume that f is (positively*) linearly homogeneous*[103]. In the economics literature, this is known as the assumption of *homothetic preferences*.[104] This assumption is not strictly justified from the viewpoint of actual economic behavior, but it leads to economic price indices that are independent from the consumer's standard of living.[105] Under this assumption, the consumer's expenditure or cost function, C(u,p) defined by (3.80) above, decomposes as follows. For positive commodity prices $p >> 0_N$ and a positive utility level u, we have by the definition of C as the minimum cost of achieving the given utility level u:

$$
\begin{aligned}
(3.85)\quad C(u,p) &\equiv \min_q \{\textstyle\sum_{i=1}^n p_i q_i : f(q_1,\ldots,q_n) \geq u\} \\
&= \min_q \{\textstyle\sum_{i=1}^n p_i q_i : (1/u)f(q_1,\ldots,q_n) \geq 1\} \quad \text{dividing by } u > 0 \\
&= \min_q \{\textstyle\sum_{i=1}^n p_i q_i : f(q_1/u,\ldots,q_n/u) \geq 1\} \quad \text{using the linear homogeneity of f} \\
&= u \min_q \{\textstyle\sum_{i=1}^n p_i q_i/u : f(q_1/u,\ldots,q_n/u) \geq 1\}
\end{aligned}
$$

---

[103] The linear homogeneity property means that f satisfies the following property: $f(\lambda q) = \lambda f(q)$ for all $\lambda > 0$ and all $q >> 0_n$. This assumption is fairly restrictive in the consumer context. It implies that each indifference curve is a radial projection of the unit utility indifference curve. It also implies that all income elasticities of demand are unity, which is contradicted by empirical evidence.

[104] More precisely, Shephard (1953) defined a homothetic function to be a monotonic transformation of a linearly homogeneous function. However, if a consumer's utility function is homothetic, we can always rescale it to be linearly homogeneous without changing consumer behavior. Hence, we simply identify the homothetic preferences assumption with the linear homogeneity assumption.

[105] This particular branch of the economic approach to index number theory is due to Shephard (1953) (1970) and Samuelson and Swamy (1974). Shephard in particular realized the importance of the homotheticity assumption in conjunction with separability assumptions in justifying the existence of subindices of the overall cost of living index. It should be noted that if the consumer's change in real income or utility between the two periods under consideration is not too large, then assuming that the consumer has homothetic preferences will lead to a true cost of living index which is very close to Laspeyres-Konüs and Paasche-Konüs true cost of living indices defined above by (3.82) and (3.83). Another way of justifying the homothetic preferences assumption is to use (3.134) below, which justifies the use of the superlative Törnqvist-Theil index P_T in the context of nonhomothetic preferences. Since P_T is usually numerically close to other superlative indices that are derived using the homothetic preferences assumption, it can be seen that the assumption of homotheticity will usually not be empirically misleading.

$= u \min_z \{\{\sum_{i=1}^n p_i z_i : f(z_1,\ldots,z_n) \geq 1\}$ letting $z_i = q_i/u$
$= u\, C(1,p)$ using definition (3.80) with u = 1
$= u\, c(p)$

where $c(p) \equiv C(1,p)$ is the *unit cost function* that is corresponds to f.[106] It can be shown that the unit cost function c(p) satisfies the same regularity conditions that f satisfied; i.e., c(p) is positive, concave and (positively) linearly homogeneous for positive price vectors.[107] Substituting (3.85) into (3.80) and using $u^t = f(q^t)$ leads to the following equations:

(3.86) $\sum_{i=1}^n p_i^t q_i^t = c(p^t) f(q^t)$ for t = 0,1.

Thus under the linear homogeneity assumption on the utility function f, observed period t expenditure on the n commodities (the left hand side of (3.86) above) is equal to the period t unit cost $c(p^t)$ of achieving one unit of utility times the period t utility level, $f(q^t)$, (the right hand side of (3.86) above). Obviously, we can identify the period t unit cost, $c(p^t)$, as the period t price level P^t and the period t level of utility, $f(q^t)$, as the period t quantity level Q^t.[108]

125. The linear homogeneity assumption on the consumer's preference function f leads to a simplification for the family of Konüs true cost of living indices, $P_K(p^0,p^1,q)$, defined by (3.81) above. Using this definition for an arbitrary reference quantity vector q, we have:

(3.87) $P_K(p^0,p^1,q) \equiv C[f(q),p^1]/C[f(q),p^0]$
$= c(p^1)f(q)/c(p^0)f(q)$ using (3.85) twice
$= c(p^1)/c(p^0)$.

Thus under the homothetic preferences assumption, the entire family of Konüs true cost of living indices collapses to a single index, $c(p^1)/c(p^0)$, the ratio of the minimum costs of achieving unit utility level when the consumer faces period 1 and 0 prices respectively. Put another way, under the homothetic preferences assumption, $P_K(p^0,p^1,q)$ is independent of the reference quantity vector q.

[106] Economists will recognize the producer theory counterpart to the result C(u,p) = uc(p): if a producer's production function f is subject to constant returns to scale, then the corresponding total cost function C(u,p) is equal to the product of the output level u times the unit cost c(p).
[107] Obviously, the utility function f determines the consumer's cost function C(u,p) as the solution to the cost minimization problem in the first line of (3.85). Then the unit cost function c(p) is defined as C(1,p). Thus f determines c. But we can also use c to determine f under appropriate regularity conditions. In the economics literature, this is known as *duality theory*. For additional material on duality theory and the properties of f and c, see Samuelson (1953), Shephard (1953) and Diewert (1974) (1993b; 107-123).
[108] There is also a producer theory interpretation of the above theory; i.e., let f be the producer's (constant returns to scale) production function, let p be a vector of input prices that the producer faces, let q be an input vector and let u = f(q) be the maximum output that can be produced using the input vector q. $C(u,p) \equiv \min_q \{ \sum_{i=1}^n p_i q_i : f(q) \geq u \}$ is the producer's cost function in this case and $c(p^t)$ can be identified as the period t input price level while $f(q^t)$ is the period t aggregate input level.

126. If we use the Konüs true cost of living index defined by the right hand side of (3.87) as our price index concept, then the corresponding implicit quantity index defined using the product test (3.3) has the following form:

$$(3.88)\quad Q(p^0,p^1,q^0,q^1) \equiv \sum_{i=1}^{n} p_i^1 q_i^1 / \{\sum_{i=1}^{n} p_i^t q_i^t\, P_K(p^0,p^1,q)\} \qquad \text{using (3.4)}$$
$$= c(p^1)f(q^1) / \{c(p^0)f(q^0)\, P_K(p^0,p^1,q)\} \qquad \text{using (3.86) twice}$$
$$= c(p^1)f(q^1) / \{c(p^0)f(q^0)[c(p^1)/c(p^0)]\} \qquad \text{using (3.87)}$$
$$= f(q^1)/f(q^0).$$

Thus under the homothetic preferences assumption, the *implicit quantity index* that corresponds to the true cost of living price index $c(p^1)/c(p^0)$ is the *utility ratio* $f(q^1)/f(q^0)$. Since the utility function is assumed to be homogeneous of degree one, this is the natural definition for a quantity index.

127. In subsequent material, we will need two additional results from economic theory: Wold's Identity and Shephard's Lemma. Wold's (1944; 69-71) (1953; 145) Identity is the following result. Assuming that the consumer satisfies the cost minimization assumptions (3.80) for periods 0 and 1 and that the utility function f is differentiable at the observed quantity vectors q^0 and q^1, it can be shown[109] that the following equations hold:

$$(3.89)\quad p_i^t / \sum_{k=1}^{n} p_k^t q_k^t = [\partial f(q^t)/\partial q_i] / \sum_{k=1}^{n} q_k^t\, \partial f(q^t)/\partial q_k\ ; \quad t = 0,1\ ; \quad k = 1,\ldots,n$$

where $\partial f(q^t)/\partial q_i$ denotes the partial derivative of the utility function f with respect to the ith quantity q_i evaluated at the period t quantity vector q^t.

128. If we make the homothetic preferences assumption and assume that the utility function is linearly homogeneous, then Wold's Identity (3.89) simplifies into the following equations which will prove to be very useful:[110]

$$(3.90)\quad p_i^t / \sum_{k=1}^{n} p_k^t q_k^t = [\partial f(q^t)/\partial q_i]/f(q^t)\ ; \quad t = 0,1\ ; \quad k = 1,\ldots,n.$$

129. Shephard's (1953; 11) Lemma is the following result. Consider the period t cost minimization problem defined by (3.80) above. If the cost function $C(u^t,p^t)$ is differentiable with respect to the components of the price vector p, then the period t quantity vector q^t is equal to the vector of first order partial derivatives of the cost function with respect to the components of p; i.e., we have

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[109] To prove this, consider the first order necessary conditions for the strictly positive vector q^t to solve the period t cost minimization problem. The conditions of Lagrange with respect to the vector of q variables are: $p^t = \lambda^t \quad f(q^t)$ where λ^t is the optimal Lagrange multiplier and $\quad f(q^t)$ is the vector of first order partial derivatives of f evaluated at q^t. Note that this system of equations is the price equals a constant times marginal utility equations that are familiar to economists. Now take the inner product of both sides of this equation with respect to the period t quantity vector q^t and solve the resulting equation for λ^t. Substitute this solution back into the vector equation $p^t = \lambda^t \quad f(q^t)$ and we obtain (3.89).

[110] Differentiate both sides of the equation $f(\lambda q) = \lambda f(q)$ with respect to λ and then evaluate the resulting equation at $\lambda = 1$. We obtain the equation $\sum_{i=1}^{n} f_i(q)q_i = f(q)$ where $f_i(q) \equiv \partial f(q)/\partial q_i$.

(3.91) $q_i^t = \partial C(u^t,p^t)/\partial p_i$; $\quad i = 1,\ldots,n$; $t = 0,1$.

To explain why (3.91) holds, consider the following argument. Because we are assuming that the observed period t quantity vector q^t solves the cost minimization problem defined by $C(u^t,p^t)$, then q^t must be feasible for this problem so we must have $f(q^t) = u^t$. Thus q^t is a feasible solution for the following cost minimization problem where the general price vector p has replaced the specific period t price vector p^t:

$$(3.92)\quad C(u^t,p) \equiv \min_q \{\sum_{i=1}^n p_iq_i : f(q_1,\ldots,q_n) \geq u^t\} \leq \sum_{i=1}^n p_iq_i^t$$

where the inequality follows from the fact that $q^t \equiv (q_1^t,\ldots,q_n^t)$ is a feasible (but usually not optimal) solution for the cost minimization problem in (3.92). Now define for each strictly positive price vector p the function g(p) as follows:

$$(3.93)\quad g(p) \equiv \sum_{i=1}^n p_iq_i^t - C(u^t,p)$$

where as usual, $p \equiv (p_1,\ldots,p_n)$. Using (3.92) and (3.93), it can be seen that g(p) is minimized (over all strictly positive price vectors p) at $p = p^t$. Thus the first order necessary conditions for minimizing a differentiable function of n variables hold, which simplify to equations (3.91).

130. If we make the homothetic preferences assumption and assume that the utility function is linearly homogeneous, then using (3.85), Shephard’s Lemma (3.91) becomes:

(3.94) $q_i^t = u^t\, \partial c(p^t)/\partial p_i$; $\quad i = 1,\ldots,n$; $t = 0,1$.

Equations (3.86) can be rewritten as follows:

(3.95) $\sum_{i=1}^n p_i^tq_i^t = c(p^t)f(q^t) = c(p^t)u^t$ $\quad$ for $t = 0,1$.

Combining equations (3.94) and (3.95), we obtain the following system of equations:

(3.96) $q_i^t / \sum_{k=1}^n p_k^tq_k^t = [\partial c(p^t)/\partial p_i] / c(p^t)$; $\quad i = 1,\ldots,n$; $t = 0,1$.

Note the symmetry of equations (3.96) with equations (3.90). It is these two sets of equations that we shall use in subsequent material.

H.3 Superlative indices: the Fisher ideal index

131. Suppose the consumer has the following utility function:

(3.97) $f(q_1,\ldots,q_n) \equiv [\sum_{i=1}^n\sum_{k=1}^n a_{ik}\, q_iq_k]^{1/2}$; $a_{ik} = a_{ki}$ for all i and k.

Differentiating f(q) defined by (3.97) with respect to q_i yields the following equations:

$$(3.98)\quad f_i(q) = (1/2)[\sum_{j=1}^n\sum_{k=1}^n a_{jk}\, q_jq_k]^{-1/2}\, 2\sum_{k=1}^n a_{ik}\, q_k\, ; \qquad i = 1,\ldots,n$$
$$= \sum_{k=1}^n a_{ik}\, q_k / f(q) \qquad \text{using (3.97)}$$

where $f_i(q) \equiv \partial f(q^t)/\partial q_i$. In order to obtain the first equation in (3.98), we need to use the symmetry conditions, $a_{ik} = a_{ki}$. Now evaluate the second equation in (3.98) at the observed period t quantity vector $q^t \equiv (q_1^t,\ldots,q_n^t)$ and divide both sides of the resulting equation by $f(q^t)$. We obtain the following equations:

$$(3.99)\quad f_i(q^t)/f(q^t) = \sum_{k=1}^n a_{ik}\, q_k^t / [f(q^t)]^2 \qquad t = 0,1\, ;\ i = 1,\ldots,n.$$

Assume cost minimizing behavior for the consumer in periods 0 and 1. Since the utility function f defined by (3.97) is linearly homogeneous and differentiable, equations (3.90) will hold. Now recall the definition of the Fisher ideal quantity index, Q_F defined by (3.14) above:

$$(3.100)\quad Q_F(p^0,p^1,q^0,q^1) = [\sum_{i=1}^n p_i^0q_i^1/\sum_{k=1}^n p_k^0q_k^0]^{1/2}\, [\sum_{i=1}^n p_i^1q_i^1/\sum_{k=1}^n p_k^1q_k^0]^{1/2}$$
$$= [\sum_{i=1}^n f_i(q^0)q_i^1/f(q^0)]^{1/2}\, [\sum_{i=1}^n p_i^1q_i^1/\sum_{k=1}^n p_k^1q_k^0]^{1/2} \qquad \text{using (3.90) for } t = 0$$
$$= [\sum_{i=1}^n f_i(q^0)q_i^1/f(q^0)]^{1/2} /[\sum_{k=1}^n p_k^1q_k^0 /\sum_{i=1}^n p_i^1q_i^1]^{1/2}$$
$$= [\sum_{i=1}^n f_i(q^0)q_i^1/f(q^0)]^{1/2} /[\sum_{i=1}^n f_i(q^1)q_i^0 /f(q^1)]^{1/2} \qquad \text{using (3.90) for } t = 1$$
$$= [\sum_{i=1}^n\sum_{k=1}^n a_{ik}\, q_k^0q_i^1/[f(q^0)]^2]^{1/2} /[\sum_{i=1}^n \sum_{k=1}^n a_{ik}\, q_k^1q_i^0 /[f(q^1)]^2]^{1/2} \text{ using (3.99)}$$
$$= [1/[f(q^0)]^2]^{1/2} / [1/[f(q^1)]^2]^{1/2} \qquad \text{using (3.98) and canceling terms}$$
$$= f(q^1)/f(q^0).$$

Thus under the assumption that the consumer engages in cost minimizing behavior during periods 0 and 1 and has preferences over the n commodities that correspond to the utility function defined by (3.97), the Fisher ideal quantity index Q_F is *exactly* equal to the true quantity index, $f(q^1)/f(q^0)$.[111]

132. As was noted in section C.2 above, the price index that corresponds to the Fisher quantity index Q_F using the product test (3.3) is the Fisher price index P_F defined by (3.12). Let c(p) be the unit cost function that corresponds to the homogeneous quadratic utility function f defined by (3.97). Then using (3.95) and (3.100), it can be seen that

$$(3.101)\quad P_F(p^0,p^1,q^0,q^1) = c(p^1)/c(p^0).$$

Thus under the assumption that the consumer engages in cost minimizing behavior during periods 0 and 1 and has preferences over the n commodities that correspond to the utility function defined by (3.97), the Fisher ideal price index P_F is exactly equal to the true price index, $c(p^1)/c(p^0)$.

133. A twice continuously differentiable function f(q) of n variables $q \equiv (q_1,\ldots,q_n)$ can provide a *second order approximation* to another such function $f^*(q)$ around the point q^* if the level and all of the first and second order partial derivatives of the two functions

[111] For the early history of this result, see Diewert (1976; 184).

coincide at q*. It can be shown[112] that the homogeneous quadratic function f defined by (3.97) can provide a second order approximation to an arbitrary f* around any (strictly positive) point q* in the class of linearly homogeneous functions. Thus the homogeneous quadratic functional form defined by (3.97) is a *flexible functional form*.[113] Diewert (1976; 117) termed an index number formula $Q_F(p^0,p^1,q^0,q^1)$ that was *exactly* equal to the true quantity index $f(q^1)/f(q^0)$ (where f is a flexible functional form) *a superlative index number formula*.[114] Equation (3.100) and the fact that the homogeneous quadratic function f defined by (3.97) is a flexible functional form shows that the Fisher ideal quantity index Q_F defined by (3.14) is a superlative index number formula. Since the Fisher ideal price index P_F also satisfies (3.101) where c(p) is the unit cost function that is generated by the homogeneous quadratic utility function, we also call P_F a superlative index number formula.

134. It is possible to show that the Fisher ideal price index is a superlative index number formula by a different route. Instead of starting with the assumption that the consumer's utility function is the homogeneous quadratic function defined by (3.97), we can start with the assumption that the consumer's unit cost function is a homogeneous quadratic.[115] Thus we suppose that the consumer has the following unit cost function:

$$(3.102)\quad c(p_1,\ldots,p_n) \equiv [\sum_{i=1}^n\sum_{k=1}^n b_{ik}\, p_ip_k]^{1/2}$$

where the parameters b_{ik} satisfy the following symmetry conditions:

$$(3.103)\quad b_{ik} = b_{ki} \quad \text{for all i and k.}$$

Differentiating c(p) defined by (3.102) with respect to p_i yields the following equations:

$$(3.104)\quad c_i(p) = (1/2)[\sum_{j=1}^n\sum_{k=1}^n b_{jk}\, p_jp_k]^{-1/2}\, 2\sum_{k=1}^n b_{ik}\, p_k\,; \qquad i = 1,\ldots,n$$
$$= \sum_{k=1}^n b_{ik}\, p_k / c(p) \qquad \text{using (3.102)}$$

where $c_i(p) \equiv \partial c(p^t)/\partial p_i$. In order to obtain the first equation in (3.104), we need to use the symmetry conditions, (3.103). Now evaluate the second equation in (3.104) at the observed period t price vector $p^t \equiv (p_1^t,\ldots,p_n^t)$ and divide both sides of the resulting equation by $c(p^t)$. We obtain the following equations:

$$(3.105)\quad c_i(p^t)/c(p^t) = \sum_{k=1}^n b_{ik}\, p_k^t / [c(p^t)]^2 \qquad t = 0,1\,;\ i = 1,\ldots,n.$$

[112] See Diewert (1976; 130) and let the parameter r equal 2.
[113] Diewert (1974; 133) introduced this term to the economics literature.
[114] Fisher (1922; 247) used the term superlative to describe the Fisher ideal price index. Thus Diewert adopted Fisher's terminology but attempted to give some precision to Fisher's definition of superlativeness. Fisher defined an index number formula to be superlative if it approximated the corresponding Fisher ideal results using his data set.
[115] Given the consumer's unit cost function c(p), Diewert (1974; 112) showed that the corresponding utility function f(q) can be defined as follows: for a strictly positive quantity vector q, $f(q) \equiv 1/ \max_p \{\sum_{i=1}^n p_iq_i : c(p) = 1\}$.

As we are assuming cost minimizing behavior for the consumer in periods 0 and 1 and since the unit cost function c defined by (3.102) is differentiable, equations (3.96) will hold. Now recall the definition of the Fisher ideal price index, P_F given by (3.12) above:

$$
\begin{aligned}
(3.106)\quad P_F(p^0,p^1,q^0,q^1) &= [\sum_{i=1}^n p_i^1 q_i^0/\sum_{k=1}^n p_k^0 q_k^0]^{1/2}\ [\sum_{i=1}^n p_i^1 q_i^1/\sum_{k=1}^n p_k^0 q_k^1]^{1/2} \\
&= [\sum_{i=1}^n p_i^1 c_i(p^0)/c(p^0)]^{1/2}\ [\sum_{i=1}^n p_i^1 q_i^1/\sum_{k=1}^n p_k^0 q_k^1]^{1/2} && \text{using (3.96) for t = 0} \\
&= [\sum_{i=1}^n p_i^1 c_i(p^0)/c(p^0)]^{1/2}\ /[\sum_{k=1}^n p_k^0 q_k^1\ /\sum_{i=1}^n p_i^1 q_i^1]^{1/2} \\
&= [\sum_{i=1}^n p_i^1 c_i(p^0)/c(p^0)]^{1/2}\ /[\sum_{i=1}^n p_i^0 c_i(p^1)/c(p^1)]^{1/2} && \text{using (3.96) for t = 1} \\
&= [\sum_{i=1}^n \sum_{k=1}^n b_{ik}\ p_k^0 p_i^1/[c(p^0)]^2]^{1/2}\ /[\sum_{i=1}^n \sum_{k=1}^n b_{ik}\ p_k^1 p_i^0/[c(p^1)]^2]^{1/2} && \text{using (3.105)} \\
&= [1/[c(p^0)]^2]^{1/2}\ /\ [1/[c(p^1)]^2]^{1/2} && \text{using (3.103) and canceling terms} \\
&= c(p^1)/c(p^0).
\end{aligned}
$$

Thus under the assumption that the consumer engages in cost minimizing behavior during periods 0 and 1 and has preferences over the n commodities that correspond to the unit cost function defined by (3.102), the Fisher ideal price index P_F is *exactly* equal to the true price index, $c(p^1)/c(p^0)$.[116]

135. Since the homogeneous quadratic unit cost function c(p) defined by (3.102) is also a flexible functional form, the fact that the Fisher ideal price index P_F exactly equals the true price index $c(p^1)/c(p^0)$ means that P_F is a *superlative index number formula*.[117]

136. Suppose that the b_{ik} coefficients in (3.102) satisfy the following restrictions:

$$(3.107)\quad b_{ik} = b_i b_k \qquad \text{for } i,k = i,\ldots,n$$

where the n numbers b_i are nonnegative. In this special case of (3.102), it can be seen that the unit cost function simplifies as follows:

$$
\begin{aligned}
(3.108)\quad c(p_1,\ldots,p_n) &\equiv [\sum_{i=1}^n \sum_{k=1}^n b_i b_k\ p_i p_k]^{1/2} \\
&= [\sum_{i=1}^n b_i\ p_i\ \sum_{k=1}^n b_k\ p_k]^{1/2} \\
&= \sum_{i=1}^n b_i\ p_i\ .
\end{aligned}
$$

Substituting (3.108) into Shephard's Lemma (3.94) yields the following expressions for the period t quantity vectors, q^t:

$$(3.109)\quad q_i^t = u^t\ \partial c(p^t)/\partial p_i = b_i\ u^t \qquad i = 1,\ldots,n\ ;\ t = 0,1.$$

Thus if the consumer has the preferences that correspond to the unit cost function defined by (3.102) where the b_{ik} satisfy the restrictions (3.107), then the period 0 and 1 quantity vectors are equal to a multiple of the vector $b \equiv (b_1,\ldots,b_n)$; i.e., $q^0 = b\ u^0$ and $q^1 = b\ u^1$. Under these assumptions, the Fisher, Paasche and Laspeyres indices, P_F, P_P and P_L, *all*

[116] This result was obtained by Diewert (1976; 133-134).

[117] Note that we have shown that the Fisher index P_F is exact for the preferences defined by (3.97) as well as the preferences that are dual to the unit cost function defined by (3.102). These two classes of preferences do not coincide in general. However, if the n by n symmetric matrix A of the a_{ik} has an inverse, then it can readily be shown that the n by n matrix B of the b_{ik} will equal A^{-1}.

coincide. However, the preferences which correspond to the unit cost function defined by (3.108) are not consistent with normal consumer behavior since they imply that the consumer will not substitute away from more expensive commodities to cheaper commodities if relative prices change going from period 0 to 1.

H.4 Quadratic mean of order r superlative indices

137. It turns out that there are many other superlative index number formulae; i.e., there exist many quantity indices $Q(p^0,p^1,q^0,q^1)$ that are exactly equal to $f(q^1)/f(q^0)$ and many price indices $P(p^0,p^1,q^0,q^1)$ that are exactly equal to $c(p^1)/c(p^0)$ where the aggregator function f or the unit cost function c is a flexible functional form. We will define two families of superlative indices below.

138. Suppose the consumer has the *following quadratic mean of order r utility function*:[118]

$$(3.110)\quad f^r(q_1,\ldots,q_n) \equiv [\textstyle\sum_{i=1}^n\sum_{k=1}^n a_{ik}\, q_i^{r/2}\, q_k^{r/2}\,]^{1/r}$$

where the parameters a_{ik} satisfy the symmetry conditions $a_{ik} = a_{ki}$ for all i and k and the parameter r satisfies the restriction $r \neq 0$. Diewert (1976; 130) showed that the utility function f^r defined by (3.110) is a flexible functional form; i.e., it can approximate an arbitrary twice continuously differentiable linearly homogeneous functional form to the second order. Note that when r = 2, f^r equals the homogeneous quadratic function defined by (3.97) above.

139. Define the *quadratic mean of order r quantity index* Q^r by:

$$(3.111)\quad Q^r(p^0,p^1,q^0,q^1) \equiv \{\textstyle\sum_{i=1}^n s_i^0\,(q_i^1/q_i^0)^{r/2}\}^{1/r}\,\{\textstyle\sum_{i=1}^n s_i^1\,(q_i^1/q_i^0)^{-r/2}\}^{-1/r}$$

where $s_i^t \equiv p_i^tq_i^t/\sum_{k=1}^n p_k^tq_k^t$ is the period t expenditure share for commodity i as usual. It can be verified that when r = 2, Q^r simplifies into Q_F, the Fisher ideal quantity index.

140. Using exactly the same techniques as were used in section H.3 above, it can be shown that Q^r is exact for the aggregator function f^r defined by (3.110); i.e., we have

$$(3.112)\quad Q^r(p^0,p^1,q^0,q^1) = f^r(q^1)/f^r(q^0).$$

Thus under the assumption that the consumer engages in cost minimizing behavior during periods 0 and 1 and has preferences over the n commodities that correspond to the utility function defined by (3.110), the quadratic mean of order r quantity index Q_F is *exactly* equal to the true quantity index, $f^r(q^1)/f^r(q^0)$.[119] Since Q^r is exact for f^r and f^r is a flexible functional form, we see that the quadratic mean of order r quantity index Q^r is a

[118] This terminology is due to Diewert (1976; 129).
[119] See Diewert (1976; 130).

superlative index for each r ≠ 0. Thus there are an infinite number of superlative quantity indices.

141. For each quantity index Q^r, we can use the product test (3.3) in order to define the corresponding *implicit quadratic mean of order r price index* P^{r*}:

$$(3.113)\quad P^{r*}(p^0,p^1,q^0,q^1) \equiv \sum_{i=1}^n p_i^1 q_i^1/\{\sum_{i=1}^n p_i^0 q_i^0\, Q^r(p^0,p^1,q^0,q^1)\}$$
$$= c^{r*}(p^1)/c^{r*}(p^0)$$

where c^{r*} is the unit cost function that corresponds to the aggregator function f^r defined by (3.110) above. For each r ≠0, the implicit quadratic mean of order r price index P^{r*} is also a superlative index.

142. When r = 2, Q^r defined by (3.111) simplifies to Q_F, the Fisher ideal quantity index and P^{r*} defined by (3.113) simplifies to P_F, the Fisher ideal price index. When r = 1, Q^r defined by (3.111) simplifies to:

$$(3.114)\quad Q^1(p^0,p^1,q^0,q^1) \equiv \{\sum_{i=1}^n s_i^0\, (q_i^1/q_i^0)^{1/2}\}/\{\sum_{i=1}^n s_i^1\, (q_i^1/q_i^0)^{-1/2}\}$$
$$= [\sum_{i=1}^n p_i^1 q_i^1/\sum_{i=1}^n p_i^0 q_i^0]\{\sum_{i=1}^n p_i^0 q_i^0 (q_i^1/q_i^0)^{1/2}\}/\{\sum_{i=1}^n p_i^1 q_i^1 (q_i^1/q_i^0)^{-1/2}\}$$
$$= [\sum_{i=1}^n p_i^1 q_i^1/\sum_{i=1}^n p_i^0 q_i^0]\{\sum_{i=1}^n p_i^0 (q_i^0 q_i^1)^{1/2}\}/\{\sum_{i=1}^n p_i^1 (q_i^0 q_i^1)^{1/2}\}$$
$$= [\sum_{i=1}^n p_i^1 q_i^1/\sum_{i=1}^n p_i^0 q_i^0]/\{\sum_{i=1}^n p_i^1 (q_i^0 q_i^1)^{1/2}/\sum_{i=1}^n p_i^0 (q_i^0 q_i^1)^{1/2}\}$$
$$= [\sum_{i=1}^n p_i^1 q_i^1/\sum_{i=1}^n p_i^0 q_i^0]/\, P_W(p^0,p^1,q^0,q^1)$$

where P_W is the *Walsh price index* defined previously by (3.19). Thus P^{1*} is equal to P_W, the *Walsh price index,* and hence it is also a superlative price index.

143. Suppose the consumer has the *following quadratic mean of order r unit cost function*:[120]

$$(3.115)\quad c^r(p_1,\ldots,p_n) \equiv [\sum_{i=1}^n \sum_{k=1}^n b_{ik}\, p_i^{r/2}\, p_k^{r/2}\,]^{1/r}$$

where the parameters b_{ik} satisfy the symmetry conditions $b_{ik} = b_{ki}$ for all i and k and the parameter r satisfies the restriction r ≠ 0. Diewert (1976; 130) showed that the unit cost function c^r defined by (3.115) is a flexible functional form; i.e., it can approximate an arbitrary twice continuously differentiable linearly homogeneous functional form to the second order. Note that when r = 2, c^r equals the homogeneous quadratic function defined by (3.102) above.

144. Define the *quadratic mean of order r price index* P^r by:

$$(3.116)\quad P^r(p^0,p^1,q^0,q^1) \equiv \{\sum_{i=1}^n s_i^0\, (p_i^1/p_i^0)^{r/2}\}^{1/r}\, \{\sum_{i=1}^n s_i^1\, (p_i^1/p_i^0)^{-r/2}\}^{-1/r}$$

[120] This terminology is due to Diewert (1976; 130). This unit cost function was first defined by Denny (1974).

where $s_i^t \equiv p_i^t q_i^t / \sum_{k=1}^n p_k^t q_k^t$ is the period t expenditure share for commodity i as usual. It can be verified that when r = 2, P^r simplifies into P_F, the Fisher ideal quantity index.

145. Using exactly the same techniques as were used in section H.3 above, it can be shown that P^r is exact for the aggregator function c^r defined by (3.115); i.e., we have

(3.117) $P^r(p^0,p^1,q^0,q^1) = c^r(p^1)/c^r(p^0)$.

Thus under the assumption that the consumer engages in cost minimizing behavior during periods 0 and 1 and has preferences over the n commodities that correspond to the unit cost function defined by (3.115), the quadratic mean of order r price index P_F is *exactly* equal to the true price index, $c^r(p^1)/c^r(p^0)$.[121] Since P^r is exact for c^r and c^r is a flexible functional form, we see that the quadratic mean of order r price index P^r is a *superlative index* for each r ≠ 0. Thus there are an infinite number of superlative price indices.

146. For each price index P^r, we can use the product test (3.3) in order to define the corresponding *implicit quadratic mean of order r quantity index* $Q^{r}*$:

(3.118) $$Q^{r}*(p^0,p^1,q^0,q^1) \equiv \sum_{i=1}^n p_i^1 q_i^1 / \{\sum_{i=1}^n p_i^0 q_i^0\, P^r(p^0,p^1,q^0,q^1)\}$$
$$= f^{r}*(p^1)/ f^{r}*(p^0)$$

where $f^{r}*$ is the aggregator function that corresponds to the unit cost function c^r defined by (3.115) above.[122] For each r ≠0, the implicit quadratic mean of order r quantity index $Q^{r}*$ is also a superlative index.

147. When r = 2, P^r defined by (3.116) simplifies to P_F, the Fisher ideal price index and $Q^{r}*$ defined by (3.118) simplifies to Q_F, the Fisher ideal quantity index. When r = 1, P^r defined by (3.116) simplifies to:

(3.119) $$P^1(p^0,p^1,q^0,q^1) \equiv \{\sum_{i=1}^n s_i^0 (p_i^1/p_i^0)^{1/2}\}/\{\sum_{i=1}^n s_i^1 (p_i^1/p_i^0)^{-1/2}\}$$
$$= [\sum_{i=1}^n p_i^1 q_i^1/\sum_{i=1}^n p_i^0 q_i^0]\{\sum_{i=1}^n p_i^0 q_i^0 (p_i^1/p_i^0)^{1/2}\}/\{\sum_{i=1}^n p_i^1 q_i^1 (p_i^1/p_i^0)^{-1/2}\}$$
$$= [\sum_{i=1}^n p_i^1 q_i^1/\sum_{i=1}^n p_i^0 q_i^0]\{\sum_{i=1}^n q_i^0 (p_i^0 p_i^1)^{1/2}\}/\{\sum_{i=1}^n q_i^1 (p_i^0 p_i^1)^{1/2}\}$$
$$= [\sum_{i=1}^n p_i^1 q_i^1/\sum_{i=1}^n p_i^0 q_i^0]/\{\sum_{i=1}^n q_i^1 (p_i^0 p_i^1)^{1/2}/\sum_{i=1}^n q_i^0 (p_i^0 p_i^1)^{1/2}\}$$
$$= [\sum_{i=1}^n p_i^1 q_i^1/\sum_{i=1}^n p_i^0 q_i^0]/Q_W(p^0,p^1,q^0,q^1)$$

where Q_W is the *Walsh quantity index* defined previously by (3.53). Thus $Q^{1}*$ is equal to Q_W, the *Walsh quantity index*, and hence it is also a superlative quantity index.

H.5 Superlative indices: The Törnqvist index

148. In this subsection, we will revert to the assumptions made on the consumer in subsection H.1 above. In particular, we do not assume that the consumer's utility function f is necessarily linearly homogeneous as in sections H.2-H.4 above.

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[121] See Diewert (1976; 133-134).

[122] The function $f^{r}*$ can be defined by using c^r as follows: $f^{r}*(q) \equiv 1/ \max_p \{\sum_{i=1}^n p_i q_i : c^r(p) = 1 \}$.

149. Before we derive our main result, we require a preliminary result. Suppose the function of n variables, $f(z_1,\ldots,z_n) \equiv f(z)$, is quadratic; i.e.,

$$(3.120)\quad f(z_1,\ldots,z_n) \equiv a_0 + \sum_{i=1}^n a_i z_i + (1/2) \sum_{i=1}^n \sum_{k=1}^n a_{ik} z_i z_k \ ; \ a_{ik} = a_{ki} \text{ for all i and k,}$$

where the a_i and the a_{ik} are constants. Let $f_i(z)$ denote the first order partial derivative of f evaluated at z with respect to the ith component of z, z_i. Let $f_{ik}(z)$ denote the second order partial derivative of f with respect to z_i and z_k. Then it is well known that the second order Taylor series approximation to a quadratic function is *exact*; i.e., if f is defined by (3.120) above, then for any two points, z^0 and z^1, we have

$$(3.121)\quad f(z^1) - f(z^0) = \sum_{i=1}^n f_i(z^0)[z_i^1 - z_i^0] + (1/2) \sum_{i=1}^n \sum_{k=1}^n f_{ik}(z^0)[z_i^1 - z_i^0][z_k^1 - z_k^0].$$

It is less well known that *an average of two first order Taylor series approximations* to a quadratic function is also *exact*; i.e., if f is defined by (3.120) above, then for any two points, z^0 and z^1, we have[123]

$$(3.122)\quad f(z^1) - f(z^0) = (1/2)\sum_{i=1}^n [f_i(z^0) + f_i(z^1)][z_i^1 - z_i^0].$$

Diewert (1976; 118) and Lau (1979) showed that equation (3.122) characterized a quadratic function and called the equation the *quadratic approximation lemma*. We will be more brief and refer to (3.122) as the *quadratic identity*.

150. We now suppose that the consumer's *cost function*,[124] C(u,p), has the following *translog functional form*:[125]

$$(3.123)\quad \ln C(u,p) \equiv a_0 + \sum_{i=1}^n a_i \ln p_i + (1/2) \sum_{i=1}^n \sum_{k=1}^n a_{ik} \ln p_i \ln p_k + b_0 \ln u + \sum_{i=1}^n b_i \ln p_i \ln u + (1/2) b_{00} [\ln u]^2$$

where ln is the natural logarithm function and the parameters a_i, a_{ik}, and b_i satisfy the following restrictions:

$$(3.124)\quad a_{ik} = a_{ki} \ ; \qquad i,k = 1,\ldots,n;$$
$$(3.125)\quad \sum_{i=1}^n a_i = 1 \ ;$$
$$(3.126)\quad \sum_{i=1}^n b_i = 0 \ ;$$
$$(3.127)\quad \sum_{k=1}^n a_{ik} = 0 \ ; \qquad i = 1,\ldots,n.$$

The parameter restrictions (3.125)-(3.127) ensure that C(u,p) defined by (3.123) is linearly homogeneous in p, a property that a cost function must have. It can be shown

[123] To prove that (3.121) and (3.122) are true, use (3.120) and substitute into the left hand sides of (3.121) and (3.122). Then calculate the partial derivatives of the quadratic function defined by (3.120) and substitute these derivatives into the right hand side of (3.121) and (3.122).
[124] The consumer's cost function was defined by (3.85) above.
[125] Christensen, Jorgenson and Lau (1971) introduced this function into the economics literature.

that the translog cost function defined by (3.123)-(3.127) can provide a second order Taylor series approximation to an arbitrary cost function.[126]

151. We assume that the consumer has preferences that correspond to the translog cost function and that the consumer engages in cost minimizing behavior during periods 0 and 1. Let p^0 and p^1 be the period 0 and 1 observed price vectors and let q^0 and q^1 be the period 0 and 1 observed quantity vectors. Thus we have:

(3.128) $C(u^0,p^0) = \sum_{i=1}^n p_i^0 q_i^0$ and $C(u^1,p^1) = \sum_{i=1}^n p_i^1 q_i^1$

where C is the translog cost function defined above. We can also apply Shephard's lemma, (3.91) above:

(3.129) $q_i^t = \partial C(u^t,p^t)/\partial p_i$; $\quad i = 1,\ldots,n$; $t = 0,1$
$= [C(u^t,p^t)/p_i^t]\, \partial \ln C(u^t,p^t)/\partial \ln p_i$.

Now use (3.128) to replace $C(u^t,p^t)$ in (3.129). After some cross multiplication, equations (3.129) become the following system of equations:

(3.130) $p_i^t q_i^t / \sum_{k=1}^n p_k^1 q_k^1 \equiv s_i^t = \partial \ln C(u^t,p^t)/\partial \ln p_i$; $\quad i = 1,\ldots,n$; $t = 0,1$ or

(3.131) $s_i^t = a_i + \sum_{k=1}^n a_{ik} \ln p_k^t + b_i \ln u^t$; $\quad i = 1,\ldots,n$; $t = 0,1$

where s_i^t is the period t expenditure share on commodity i and (3.131) follows from (3.130) by differentiating (3.123) with respect to $\ln p_i$.

152. Define the geometric average of the period 0 and 1 utility levels as u*; i.e., define

(3.132) $u^* \equiv [u^0 u^1]^{1/2}$.

Now observe that the right hand side of the equation that defines the natural logarithm of the translog cost function, equation (3.123), is a quadratic function of the variables $z_i \equiv \ln p_i$ if we hold utility constant at the level u*. Hence we can apply the quadratic identity, (3.122), and get the following equation:

(3.133) $\ln C(u^*,p^1) - \ln C(u^*,p^0)$
$= (1/2)\sum_{i=1}^n [\partial \ln C(u^*,p^0)/\partial \ln p_i + \partial \ln C(u^*,p^1)/\partial \ln p_i][\ln p_i^1 - \ln p_i^0]$
$= (1/2)\sum_{i=1}^n [a_i+\sum_{k=1}^n a_{ik} \ln p_k^0 + b_i \ln u^* + a_i+\sum_{k=1}^n a_{ik} \ln p_k^1 + b_i \ln u^*][\ln p_i^1 - \ln p_i^0]$
differentiating (3.123) at the points (u^*,p^0) and (u^*,p^1)
$= (1/2)\sum_{i=1}^n [a_i+\sum_{k=1}^n a_{ik} \ln p_k^0 + b_i \ln[u^0u^1]^{1/2} + a_i+\sum_{k=1}^n a_{ik} \ln p_k^1 + b_i \ln[u^0u^1]^{1/2}][\ln p_i^1 - \ln p_i^0]$
using definition (3.132) for u*
$= (1/2)\sum_{i=1}^n [a_i+\sum_{k=1}^n a_{ik} \ln p_k^0 + b_i \ln u^0 + a_i+\sum_{k=1}^n a_{ik} \ln p_k^1 + b_i \ln u^1][\ln p_i^1 - \ln p_i^0]$

---

[126] It can also be shown that if all of the $b_i = 0$ and $b_{00} = 0$, then $C(u,p) = uC(1,p) \equiv uc(p)$; i.e., with these additional restrictions on the parameters of the general translog cost function, we have homothetic preferences. Note that we also assume that utility u is scaled so that u is always positive.

rearranging terms

$= (1/2)\sum_{i=1}^{n} [\partial \ln C(u^0,p^0)/\partial \ln p_i + \partial \ln C(u^1,p^1)/\partial \ln p_i][\ln p_i^1 - \ln p_i^0]$

differentiating (3.123) at the points (u^0,p^0) and (u^1,p^1)

$= (1/2)\sum_{i=1}^{n} [s_i^0 + s_i^1][\ln p_i^1 - \ln p_i^0]$ using (3.131).

The last equation in (3.133) can be recognized as the logarithm of the Törnqvist-Theil index number formula P_T defined earlier by (3.66). Hence exponentiating both sides of (3.133) yields the following equality between the true cost of living between periods 0 and 1, evaluated at the intermediate utility level u* and the observable Törnqvist-Theil index P_T:[127]

(3.134) $C(u^*,p^1)/C(u^*,p^0) = P_T(p^0,p^1,q^0,q^1).$

Since the translog cost function which appears on the left hand side of (3.134) is a flexible functional form, the Törnqvist-Theil price index P_T is also a *superlative index*.

153. It is somewhat mysterious how a ratio of *unobservable* cost functions of the form appearing on the left hand side of the above equation can be *exactly* estimated by an *observable* index number formula but the key to this mystery is the assumption of cost minimizing behavior and the quadratic identity (3.122) along with the fact that derivatives of cost functions are equal to quantities, as specified by Shephard's lemma, (3.91). In fact, all of the exact index number results derived in sections H.3 and H.4 can be derived using transformations of the quadratic identity along with Shephard's lemma (or Wold's identity, (3.98) above).[128] Fortunately, for most empirical applications, assuming that the consumer has (transformed) quadratic preferences will be an adequate assumption so the results presented in section H.3-H.5 are quite useful to index number practitioners who are willing to adopt the economic approach to index number theory.[129] Essentially, the economic approach to index number theory provides a strong justification for the use of the Fisher price index P_F defined by (3.12), the Törnqvist-Theil price index P_T defined by (3.66), the implicit quadratic mean of order r price indices P^{r*} defined by (3.113) (when r = 1, this index is the Walsh price index defined by (3.19) above) and the quadratic mean of order r price indices P^r defined by (3.116). In the next section, we ask if it matters which one of these formula is chosen as "best".

H.6 The approximation properties of superlative indices

[127] This result is due to Diewert (1976; 122).

[128] See Diewert (2000b). Wold's identity says that derivatives of the utility function are proportional to prices.

[129] However, if consumer preferences are nonhomothetic and the change in utility is substantial between the two situations being compared, then we may want to compute separately the Laspeyres-Konüs and Paasche-Konüs true cost of living indices defined above by (3.82) and (3.83), $C(u^0,p^1)/C(u^0,p^0)$ and $C(u^1,p^1)/C(u^1,p^0)$ respectively. In order to do this, we would have to use econometrics and estimate empirically the consumer's cost or expenditure function. However, if we are willing to make the assumption that the consumer's cost function can be adequately represented by a general translog cost function, then we can use (3.134) to calculate the true index $C(u^*,p^1)/C(u^*,p^0)$ where $u^* \equiv [u^0u^1]^{1/2}$.

154. The results of sections H.3-H.5 provide us with a large number of index number formulae which appear to be equally good from the viewpoint of the economic approach to index number theory. Two questions arise as a consequence of these results:

- Does it matter which of these formulae is chosen?
- If it does matter, which formula should be chosen?

155. With respect to the first question, Diewert (1978; 888) showed that all of the superlative index number formulae listed above in sections H.3-H.5 approximate each other to the second order around any point where the two price vectors, p^0 and p^1,are equal and where the two quantity vectors, q^0 and q^1, are equal. In particular, this means that we have the following equalities for all r and s not equal to 0 provided that $p^0 = p^1$ and $q^0 = q^1$:[130]

$$(3.135)\quad P_T(p^0,p^1,q^0,q^1) = P^r(p^0,p^1,q^0,q^1) = P^{s*}(p^0,p^1,q^0,q^1);$$
$$(3.136)\quad \partial P_T(p^0,p^1,q^0,q^1)/\partial p_i^t = \partial P^r(p^0,p^1,q^0,q^1)/\partial p_i^t = \partial P^{s*}(p^0,p^1,q^0,q^1)/\partial p_i^t\ ; \quad i = 1,\ldots,n;\ t = 0,1;$$
$$(3.137)\quad \partial P_T(p^0,p^1,q^0,q^1)/\partial q_i^t = \partial P^r(p^0,p^1,q^0,q^1)/\partial q_i^t = \partial P^{s*}(p^0,p^1,q^0,q^1)/\partial q_i^t\ ; \quad i = 1,\ldots,n;\ t = 0,1;$$
$$(3.138)\quad \partial^2 P_T(p^0,p^1,q^0,q^1)/\partial p_i^t\partial p_k^t = \partial^2 P^r(p^0,p^1,q^0,q^1)/\partial p_i^t\partial p_k^t = \partial^2 P^{s*}(p^0,p^1,q^0,q^1)/\partial p_i^t\partial p_k^t\ ; \quad i,k = 1,\ldots,n;\ t = 0,1;$$
$$(3.139)\quad \partial^2 P_T(p^0,p^1,q^0,q^1)/\partial p_i^t\partial q_k^t = \partial^2 P^r(p^0,p^1,q^0,q^1)/\partial p_i^t\partial q_k^t = \partial^2 P^{s*}(p^0,p^1,q^0,q^1)/\partial p_i^t\partial q_k^t\ ; \quad i,k = 1,\ldots,n;\ t = 0,1;$$
$$(3.140)\quad \partial^2 P_T(p^0,p^1,q^0,q^1)/\partial q_i^t\partial q_k^t = \partial^2 P^r(p^0,p^1,q^0,q^1)/\partial q_i^t\partial q_k^t = \partial^2 P^{s*}(p^0,p^1,q^0,q^1)/\partial q_i^t\partial q_k^t\ ; \quad i,k = 1,\ldots,n;\ t = 0,1;$$

where the Törnqvist-Theil price index P_T is defined by (3.66), the implicit quadratic mean of order r price indices P^{s*} are defined by (3.113) and the quadratic mean of order r price indices P^r are defined by (3.116). Using the above results, Diewert (1978; 884) concluded that "all superlative indices closely approximate each other".

156. However, the above conclusion is not true even though the equations (3.135)-(3.140) are true. The problem is that the quadratic mean of order r price indices P^r and the implicit quadratic mean of order s price indices P^{s*} are (continuous) functions of the parameters r and s respectively. Hence as r and s become very large in magnitude, the indices P^r and P^{s*} can differ substantially from say $P^2 = P_F$, the Fisher ideal index . In fact, using definition (3.116) and the limiting properties of means of order r[131], Robert Hill (2000;7) showed that P^r has the following limit as r approaches plus or minus infinity:

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[130] To prove the equalities in (3.136)-(3.140), simply differentiate the various index number formulae and evaluate the derivatives at $p^0 = p^1$ and $q^0 = q^1$. Actually, equations (3.135)-(3.140) are still true provided that $p^1 = \lambda p^0$ and $q^1 = \mu q^0$ for any numbers $\lambda > 0$ and $\mu > 0$; i.e., provided that the period 1 price vector is proportional to the period 0 price vector and that the period 1 quantity vector is proportional to the period 0 quantity vector.

[131] See Hardy, Littlewood and Polya (1934).

$$(3.141)\quad \lim_{r\to+\infty} P^r(p^0,p^1,q^0,q^1) = \lim_{r\to-\infty} P^r(p^0,p^1,q^0,q^1) = [\min_i\{p_i^1/p_i^0\}\max_i\{p_i^1/p_i^0\}]^{1/2} .$$

Using Hill's method of analysis, it can be shown that the implicit quadratic mean of order r price index has the following limit as r approaches plus or minus infinity:

$$(3.142)\quad \lim_{r\to+\infty} P^{r*}(p^0,p^1,q^0,q^1) = \lim_{r\to-\infty} P^{r*}(p^0,p^1,q^0,q^1) = \sum_{i=1}^n p_i^1q_i^1/ \sum_{i=1}^n p_i^0q_i^0\, [\min_i\{q_i^1/q_i^0\}\max_i\{q_i^1/q_i^0\}]^{1/2} .$$

Thus for r large in magnitude, P^r and P^{r*} can differ substantially from P_T, P^1, $P^{1*} = P_W$ (the Walsh price index) and $P^2 = P^{2*} = P_F$ (the Fisher ideal index).[132]

157. Although Robert Hill's theoretical and empirical results demonstrate conclusively that all superlative indices will not necessarily closely approximate each other, there is still the question of how well the more commonly used superlative indices will approximate each other. All of the commonly used superlative indices, P^r and P^{r*}, fall into the interval $0 \le r \le 2$.[133] Robert Hill (2000; 16) summarized how far apart the Törnqvist and Fisher indices were making all possible bilateral comparisons between any two data points for his time series data set as follows:

"The superlative spread S(0,2) is also of interest since, in practice, Törnqvist (r = 0)and Fisher (r = 2) are by far the two most widely used superlative indexes. In all 153 bilateral comparisons, S(0,2) is less than the Paasche-Laspeyres spread and on average, the superlative spread is only 0.1 percent. It is because attention, until now, has focussed almost exclusively on superlative indexes in the range $0 \le r \le 2$ that a general misperception has persisted in the index number literature that all superlative indexes approximate each other closely."

Thus for Hill's time series data set covering 64 components of U.S. GDP from 1977 to 1994 and making all possible bilateral comparisons between any two years, the Fisher and Törnqvist price indices differed by only 0.1 percent on average. This close correspondence is consistent with the results of other empirical studies using annual time series data.[134] Additional evidence on this topic may be found in Appendix 3.3 below.

158. We have found that several index number formulae seem "best" when viewed from various perspectives. Thus we found that the Fisher ideal index $P_F = P^2 = P^{2*}$ defined by (3.12) seemed to be best from the axiomatic viewpoint, the Törnqvist-Theil price index P_T defined by (3.66) seemed to be best from the stochastic viewpoint, and the Walsh index P_W defined by (3.19) (which is equal to the implicit quadratic mean of order r price indices P^{r*} defined by (3.113) when r = 1) seemed to be best from the viewpoint of the "pure" price index. The results presented in this section indicate that for "normal" time series data, these 3 indices will give virtually the same answer. To determine precisely which one of these three alternative indices to use as a theoretical target or

[132] Robert Hill (2000) documents this for two data sets. His time series data consists of annual expenditure and quantity data for 64 components of U.S. GDP from 1977 to 1994. For this data set, Hill (2000; 16) found that "superlative indexes can differ by more than a factor of two (i.e., by more than 100 percent), even though Fisher and Törnqvist never differ by more than 0.6 percent."

[133] Diewert (1980; 451) showed that the Törnqvist index P_T is a limiting case of P^r as r tends to 0.

[134] See for example Diewert (1978; 894) or Fisher (1922), which is reproduced in Diewert (1976; 135).

actual index, the statistical agency will have to decide which approach to bilateral index number theory is most consistent with its goals.

H.7 Superlative indices and two stage aggregation

159. Most statistical agencies use the Laspeyres formula to aggregate prices in two stages. At the first stage of aggregation, the Laspeyres formula is used to aggregate components of the overall index (e.g., food, clothing, services, etc.) and then at the second stage of aggregation, these component subindices are further combined into the overall index. The following question then naturally arises: does the index computed in two stages coincide with the index computed in a single stage? We address this question initially in the context of the Laspeyres formula.[135]

160. We now suppose that the price and quantity data for period t, p^t and q^t, can be written in terms of M subvectors as follows:

$$(3.143) \quad p^t = (p^{t1}, p^{t2}, \ldots ,p^{tM}) \;;\; q^t = (q^{t1}, q^{t2}, \ldots ,q^{tM}) \;;\; t = 0,1$$

where the dimensionality of the subvectors p^{tm} and q^{tm} is N_m for m = 1,2,…,M with the sum of the dimensions N_m equal to n. These subvectors correspond to the price and quantity data for subcomponents of the consumer price index for period t. We construct subindices for each of these components going from period 0 to 1. For the base period, we set the price for each of these subcomponents, say P_m^0 for m = 1,2,…M, equal to 1 and we set the corresponding base period subcomponent quantities, say Q_m^0 for m = 1,2,…,M, equal to the base period value of consumption for that subcomponent for m = 1,2,…,M; i.e., we have:

$$(3.144) \quad P_m^0 \equiv 1 \;;\; Q_m^0 \equiv \textstyle\sum_{i=1}^{Nm} p_i^{0m} q_i^{0m} \qquad \text{for } m = 1,2,\ldots,M.$$

Now we use the Laspeyres formula in order to construct a period 1 price for each subcomponent, say P_m^1 for m = 1,2,…,M, of the consumer price index. Since the dimensionality of the subcomponent vectors, p^{tm} and q^{tm} , differ from the dimensionality of the complete period t vectors of prices and quantities, p^t and q^t , we shall use different symbols for these subcomponent Laspeyres indices, say P_L^m for m = 1,2,…M. Thus the period 1 subcomponent prices are defined as follows:

$$(3.145) \quad P_m^1 \equiv P_L^m(p^{0m},p^{1m},q^{0m},q^{1m}) \equiv \textstyle\sum_{i=1}^{Nm} p_i^{1m} q_i^{0m} / \sum_{i=1}^{Nm} p_i^{0m} q_i^{0m} \quad \text{for } m = 1,2,\ldots M.$$

Once the period 1 prices for the M subindices have been defined by (3.145), then corresponding subcomponent period 1 quantities Q_m^1 for m = 1,2,…,M can be defined by deflating the period 1 subcomponent values $\sum_{i=1}^{Nm} p_i^{1m} q_i^{1m}$ by the prices P_m^1 defined by (3.145); i.e., we have:

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[135] Much of the initial material in this section is adapted from Diewert (1978) and Alterman, Diewert and Feenstra (1999). See also Balk (1996b) for a discussion of alternative definitions for the two stage aggregation concept and references to the literature on this topic.

(3.146) $Q_m^1 \equiv \sum_{i=1}^{Nm} p_i^{1m} q_i^{1m} / P_m^1$ for m = 1,2,…,M.

We can now define subcomponent price and quantity vectors for each period t = 0,1 using equations (3.144) to (3.146) above. Thus we define the period 0 and 1 subcomponent price vectors P^0 and P^1 as follows:

(3.147) $P^0 = (P_1^0, P_2^0,\ldots,P_M^0) \equiv 1_M$; $P^1 = (P_1^1, P_2^1,\ldots,P_M^1)$

where 1_M denotes a vector of ones of dimension M and the components of P^1 are defined by (3.145). The period 0 and 1 subcomponent quantity vectors Q^0 and Q^1 are defined as follows:

(3.148) $Q^0 = (Q_1^0, Q_2^0,\ldots,Q_M^0)$; $Q^1 = (Q_1^1, Q_2^1,\ldots,Q_M^1)$

where the components of Q^0 are defined in (3.144) and the components of Q^1 are defined by (3.146). The price and quantity vectors in (3.147) and (3.148) represent the results of the first stage aggregation. We can now use these vectors as inputs into the second stage aggregation problem; i.e., we can now apply the Laspeyres price index formula using the information in (3.147) and (3.148) as inputs into the index number formula. Since the price and quantity vectors that are inputs into this second stage aggregation problem have dimension M instead of the single stage formula which utilized vectors of dimension n, we need a different symbol for our new Laspeyres index which we choose to be P_L*. Thus the Laspeyres price index computed in two stages can be denoted as $P_L*(P^0,P^1,Q^0,Q^1)$. We ask whether this two stage Laspeyres index equals the corresponding single stage index P_L that we have studied in the previous sections of this chapter; i.e., we ask whether

(3.149) $P_L*(P^0,P^1,Q^0,Q^1) = P_L(p^0,p^1,q^0,q^1)$.

If the Laspeyres formula is used at each stage of each aggregation, the answer to the above question is yes: straightforward calculations show that the Laspeyres index calculated in two stages equals the Laspeyres index calculated in one stage.

161. Now suppose we use the Fisher or Törnqvist formula at each stage of the aggregation; i.e., in equations (3.145), suppose we replace the Laspeyres formula $P_L^m(p^{0m},p^{1m},q^{0m},q^{1m})$ by the Fisher formula $P_F^m(p^{0m},p^{1m},q^{0m},q^{1m})$ (or by the Törnqvist formula $P_T^m(p^{0m},p^{1m},q^{0m},q^{1m})$) and in equation (3.149), we replace $P_L*(P^0,P^1,Q^0,Q^1)$ by P_F* (or by P_T*) and replace $P_L(p^0,p^1,q^0,q^1)$ by P_F (or by P_T). Then do we obtain counterparts to the two stage aggregation result for the Laspeyres formula, (3.149)? The answer is no; it can be shown that, in general,

(3.150) $P_F*(P^0,P^1,Q^0,Q^1) \neq P_F(p^0,p^1,q^0,q^1)$ and $P_T*(P^0,P^1,Q^0,Q^1) \neq P_T(p^0,p^1,q^0,q^1)$.

Similarly, it can be shown that the quadratic mean of order r index number formula P^r defined by (3.116) and the implicit quadratic mean of order r index number formula P^{r}* defined by (3.113) are also not consistent in aggregation.

162. However, even though the Fisher and Törnqvist formulae are not *exactly* consistent in aggregation, it can be shown that these formulae are *approximately* consistent in aggregation. More specifically, it can be shown that the two stage Fisher formula P_F* and the single stage Fisher formula P_F in (3.150), both regarded as functions of the 4n variables in the vectors p^0,p^1,q^0,q^1, approximate each other to the second order around a point where the two price vectors are equal (so that $p^0 = p^1$) and where the two quantity vectors are equal (so that $q^0 = q^1$) and a similar result holds for the two stage and single stage Törnqvist indices in (3.150).[136] As we saw in the previous section, the single stage Fisher and Törnqvist indices have a similar approximation property so all four indices in (3.150) approximate each other to the second order around an equal (or proportional) price and quantity point. Thus for normal time series data, single stage and two stage Fisher and Törnqvist indices will usually be numerically very close.[137] We illustrate this result in Appendix 3.3 for an artificial data set.

163. Similar approximate consistency in aggregation results (to the results for the Fisher and Törnqvist formulae explained in the previous paragraph) can be derived for the *quadratic mean of order r indices*, P^r, and for the implicit quadratic mean of order r indices, P^{r}*; see Diewert (1978; 889). However, the results of Hill (2000) again imply that *the second order approximation property of the single stage quadratic mean of order r index P^r to its two stage counterpart will break down as r approaches either plus or minus infinity*. To see this, consider a simple example where there are only four commodities in total. Let the first price ratio $p_1^{\,1}/p_1^{\,0}$ be equal to the positive number a, let the second two price ratios $p_i^{\,1}/p_i^{\,0}$ equal b and let the last price ratio $p_4^{\,1}/p_4^{\,1}$ equal c where we assume a < c and a ≤ b ≤ c. Using Hill's result (3.141), the limiting value of the single stage index is:

$$(3.151)\quad \lim_{r\to+\infty} P^r(p^0,p^1,q^0,q^1) = \lim_{r\to-\infty} P^r(p^0,p^1,q^0,q^1) = [\min_i\{p_i^{\,1}/p_i^{\,0}\}\max_i\{p_i^{\,1}/p_i^{\,0}\}]^{1/2} = [ac]^{1/2}.$$

Now let us aggregate commodities 1 and 2 into a subaggregate and commodities 3 and 4 into another subaggregate. Using Hill's result (3.141) again, we find that the limiting price index for the first subaggregate is $[ab]^{1/2}$ and the limiting price index for the second subaggregate is $[bc]^{1/2}$. Now apply the second stage of aggregation and use Hill's result once again to conclude that the limiting value of the two stage aggregation using P^r as our index number formula is $[ab^2c]^{1/4}$. Thus the limiting value as r tends to plus or minus infinity of the single stage aggregate over the two stage aggregate is $[ac]^{1/2}/[ab^2c]^{1/4} = [ac/b^2]^{1/4}$. Now b can take on any value between a and c and so the ratio of the single stage limiting P^r to its two stage counterpart can take on any value between $[c/a]^{1/4}$ and

136 See Diewert (1978; 889). In other words, a string of equalities similar to (3.135)-(3.140) hold between the two stage indices and their single stage counterparts. In fact, these equalities are still true provided that $p^1 = \lambda p^0$ and $q^1 = \mu q^0$ for any numbers $\lambda > 0$ and $\mu > 0$.

137 For an empirical comparison of the four indices, see Diewert (1978; 894-895). For the Canadian consumer data considered there, the chained two stage Fisher in 1971 was 2.3228 and the corresponding chained two stage Törnqvist was 2.3230, the same values as for the corresponding single stage indices.

$[a/c]^{1/4}$. Since c/a is less than 1 and a/c is greater than 1, it can be seen that the ratio of the single stage to the two stage index can be arbitrarily far from 1 as r becomes large in magnitude with an appropriate choice of the numbers a, b and c.

164. The results in the previous paragraph show that we must be cautious in assuming that *all* superlative indices will be approximately consistent in aggregation. However, for the three most commonly used superlative indices (the Fisher ideal P_F, the Törnqvist-Theil P_T and the Walsh P_W), the available empirical evidence indicates that these indices satisfy the consistency in aggregation property to a sufficiently high enough degree of approximation that users will not be unduly troubled by any inconsistencies.[138]

H.8 The Lloyd-Moulton index number formula.

165. The final index number formula that we wish to discuss in this section on the single household economic approach to index number theory is a potentially very useful one for statistical agencies that are faced with the problem of producing a CPI in a timely manner. The index number formula that we will discuss makes use of the same information that is required in order to implement a Laspeyres index except one additional piece of information is required.

166. In this section, we make the same assumptions on the consumer that we made in section H.2 above. In particular, we assume that the consumer's utility function f(q) is linearly homogeneous[139] and the corresponding unit cost function is c(p). We suppose that the unit cost function has the following functional form:

(3.152) $c(p) \equiv \alpha_0 \,[\Sigma_{i=1}^{n} \alpha_i\, p_i^{1-\sigma}]^{1/(1-\sigma)}$ if $\sigma \neq 1$ or
$\ln c(p) \equiv \alpha_0 + \Sigma_{i=1}^{n} \alpha_i \ln p_i$ if $\sigma = 1$

where the α_i and σ are nonnegative parameters with $\sum_{i=1}^{n} \alpha_i = 1$. The unit cost function defined by (3.152) corresponds to *a Constant Elasticity of Substitution (CES) aggregator function* which was introduced into the economics literature by Arrow, Chenery, Minhas and Solow (1961)[140]. The parameter σ is the *elasticity of substitution* ; when $\sigma = 0$, the unit cost function defined by (3.152) becomes linear in prices and hence corresponds to a fixed coefficients aggregator function which exhibits 0 substitutability between all commodities. When $\sigma = 1$,the corresponding aggregator function is a Cobb-Douglas function. When σ approaches $+\infty$, the corresponding aggregator function f approaches a linear aggregator function which exhibits infinite substitutability between each pair of inputs. The CES unit cost function defined by (3.152) is *not* a fully flexible functional form (unless the number of commodities n being aggregated is 2) but it is considerably more flexible than the zero substitutability aggregator function that is exact for the Laspeyres and Paasche price indices.

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138 See Appendix 3.3 for some additional evidence on this topic.
139 Thus we are assuming homothetic preferences in this section.
140 In the mathematics literature, this aggregator function is known as a mean of order r; see Hardy, Littlewood and Polyá (1934; 12-13).

167. Under the assumption of cost minimizing behavior in period 0, Shephard's Lemma, (3.94) above, tells us that the observed first period consumption of commodity i, q_i^0, will be equal to $u^0\, \partial c(p^0)/\partial p_i$ where $\partial c(p^0)/\partial p_i$ is the first order partial derivative of the unit cost function with respect to the ith commodity price evaluated at the period 0 prices and $u^0 = f(q^0)$ is the aggregate (unobservable) level of period 0 utility. Using the CES functional form defined by (3.152) and assuming that $\sigma \neq 1$, we obtain the following equations:

$$(3.153)\quad \begin{aligned} q_i^0 &= u^0\, \alpha_0\, [\Sigma_{k=1}^n\, \alpha_k\, (p_k^0)^r\,]^{(1/r)\,-1}\, \alpha_i\, (p_i^0)^{r\,-1}\ ; \quad && r \equiv 1 - \sigma \neq 0\ ;\quad i = 1,2,\ldots,n \\ &= u^0 c(p^0)\, \alpha_i\, (p_i^0)^{r\,-1} / [\Sigma_{k=1}^n\, \alpha_k\, (p_k^0)^r\,] && \text{using (3.152).} \end{aligned}$$

Equations (3.153) can be rewritten as

$$(3.154)\quad p_i^0 q_i^0 / u^0 c(p^0) = \alpha_i\, (p_i^0)^r / \Sigma_{k=1}^n\, \alpha_k\, (p_k^0)^r\ ; \qquad i = 1,2,\ldots,n$$

where $r \equiv 1 - \sigma$. Now consider the following *Lloyd (1975) Moulton (1996) index number formula*:

$$(3.155)\quad P_{LM}(p^0,p^1,q^0,q^1) \equiv [\Sigma_{i=1}^n\, s_i^0\, (p_i^1/p_i^0)^{1-\sigma}\,]^{1/(1-\sigma)}\ ; \qquad \sigma \neq 1$$

where s_i^0 is the period 0 expenditure share of commodity i as usual; i.e., we have

$$(3.156)\quad \begin{aligned} s_i^0 &\equiv p_i^0 q_i^0 / \textstyle\sum_{k=1}^n p_k^0 q_k^0\ ; && i = 1,2,\ldots,n \\ &= p_i^0 q_i^0 / u^0\, c(p^0) && \text{using the assumption of cost minimizing behavior} \\ &= \alpha_i\, (p_i^0)^r / \Sigma_{k=1}^n\, \alpha_k\, (p_k^0)^r && \text{using (3.154).} \end{aligned}$$

If we substitute (3.156) into (3.155), we find that:

$$(3.157)\quad \begin{aligned} P_{LM}(p^0,p^1,q^0,q^1) &= [\Sigma_{i=1}^n\, s_i^0\, (p_i^1/p_i^0)^r\,]^{1/r} \\ &= [\Sigma_{i=1}^n\, \{\alpha_i\, (p_i^0)^r / \Sigma_{k=1}^n\, \alpha_k\, (p_k^0)^r\}\, (p_i^1/p_i^0)^r\,]^{1/r} \\ &= [\Sigma_{i=1}^n\, \alpha_i\, (p_i^1)^r / \Sigma_{k=1}^n\, \alpha_k\, (p_k^0)^r\,]^{1/r} \\ &= \alpha_0\, [\Sigma_{i=1}^n\, \alpha_i\, (p_i^1)^r\,]^{1/r} / \alpha_0\, [\Sigma_{k=1}^n\, \alpha_k\, (p_k^0)^r\,]^{1/r} \\ &= c(p^1)/c(p^0) \quad \text{using } r \equiv 1 - \sigma \text{ and definition (3.152).} \end{aligned}$$

168. Equation (3.157) shows that the Lloyd Moulton index number formula P_{LM} is *exact* for CES preferences. Lloyd (1975) and Moulton (1996) independently derived this result but it was Moulton who appreciated the significance of the formula (3.155). Note that in order to evaluate (3.155) numerically, we require information on:

- base period expenditure shares s_i^0 ;
- the price relatives p_i^1/p_i^0 between the base period and the current period and
- an estimate of the elasticity of substitution between the commodities in the aggregate, σ.

The first two pieces of information are the standard information sets that statistical agencies use to evaluate the Laspeyres price index P_L (note that P_{LM} reduces to P_L if $\sigma = 0$). Hence, if the statistical agency is able to estimate the elasticity of substitution σ based on past experience[141], *then the Lloyd Moulton price index can be evaluated using essentially the same information set that is used in order to evaluate the traditional Laspeyres index*. Moreover, the resulting consumer price index will be free of *substitution bias* to a reasonable degree of approximation.[142] Of course, the practical problem with implementing this methodology is that estimates of the elasticity of substitution parameter σ are bound to be somewhat uncertain and hence the resulting Lloyd Moulton index may be subject to charges that it is not *objective* or *reproducible*. The statistical agency will have to balance the benefits of reducing substitution bias with these possible costs.

I. Economic approaches: The case of many households

I.1 Plutocratic cost of living indices and observable bounds

169. Up to this point, we have implicitly assumed a representative consumer model. In this section[143], we consider some of the problems involved in the construction of a superlative index when there are many households or regions in the economy and our goal is the production of a national index. In our algebra below, we allow for an arbitrary number of households, H say, so in principle, each household in the economy under consideration could have its own consumer price index. However, in practice, it will be necessary to group households into various classes and within each class, it will be necessary to assume that the group of households in the class behaves as if it were a single household in order to apply the economic approach to index number theory. Our partition of the economy into H household classes can also be given a regional interpretation: each household class can be interpreted as a group of households within a region of the country under consideration.

170. In this subsection, we will consider an economic approach to the CPI that is based on the *plutocratic cost of living index* that was originally defined by Prais (1959). This concept was further refined by Pollak (1980; 276) (1981; 328) who defined his *Scitovsky-Laspeyres cost of living index* as the ratio of total expenditure required to enable each

[141] For the first application of this methodology (in the context of the consumer price index), see Shapiro and Wilcox (1997; 121-123). They calculated superlative Törnqvist indices for the U.S. for the years 1986-1995 and then calculated the Lloyd Moulton CES index for the same period using various values of σ. They then chose the value of σ (which was .7) which caused the CES index to most closely approximate the Törnqvist index. Essentially the same methodology was used by Alterman, Diewert and Feenstra (1999) in their study of U.S. import and export price indices. For alternative methods for estimating σ, see Balk (2000b).

[142] What is a "reasonable" degree of approximation depends on the context. Assuming that consumers have C.E.S. preferences is not a reasonable assumption in the context of estimating elasticities of demand: we require at least a second order approximation to the consumer's preferences in this context. However, in the context of approximating changes in a consumer's expenditures on the n commodities under consideration, it is usually adequate to assume a C.E.S. approximation.

[143] Much of the material in this section is based on Diewert (1983a) (2000a) (2001).

household in the economy under consideration to attain its base period indifference surface at period 1 prices to that required at period 0 prices. In the following paragraph, we will make various assumptions and explain this concept more fully.

171. Suppose that there are H households (or regions) in the economy and suppose further that there are n commodities in the economy in periods 0 and 1 that households consume *and* that we wish to include in our definition of the cost of living. Denote an n dimensional vector of commodity consumption in a given period by $q \equiv (q_1,q_2,\ldots, q_n)$ as usual. Denote the vector of period t market prices faced by household h by $p_h^t \equiv (p_{h1}^t,p_{h2}^t,\ldots,p_{hN}^t)$ for t = 0,1. Note that we are *not* assuming that each household faces the same vector of commodity prices. In addition to the market commodities that are in the vector q, we assume that each household is affected by an M dimensional vector of *environmental*[144] *or demographic*[145] *variables or public goods*, $e \equiv (e_1,e_2,\ldots,e_M)$. We suppose that there are H households (or regions) in the economy during periods 0 and 1 and the preferences of household h over different combinations of market commodities q and environmental variables e can be represented by the continuous utility function $f^h(q,e)$ for $h = 1,2,\ldots,H$.[146] For periods t = 0,1 and for households $h = 1,2,\ldots,H$, it is assumed that the observed household h consumption vector $q_h^t \equiv (q_{h1}^t,\ldots,q_{hN}^t)$ is a solution to the following household h expenditure minimization problem:

$$(3.158)\quad \min_q \{ p_h^t\cdot q : f^h(q,e_h^t) \geq u_h^t \} \equiv C^h(u_h^t,e_h^t, p_h^t) \; ; \; t = 0,1; \; h = 1,2,\ldots H$$

where e_h^t is the environmental vector facing household h in period t, $u_h^t \equiv f^h(q_h^t,e_h^t)$ is the utility level achieved by household h during period t and C^h is the cost or expenditure function that is dual to the utility function f^h.[147] Basically, these assumptions mean that each household has *stable preferences* over the same list of commodities during the two periods under consideration, the same households appear in each period and each household chooses its consumption bundle in the most cost efficient way during each period, conditional on the environmental vector that it faces during each period. Note again that the household (or regional) prices are in general different across households (or regions).

172. With the above assumptions in mind, we generalize Pollak (1980) (1981) and Diewert (1983a; 190)[148] and define the class of *conditional plutocratic cost of living indices*, $P^*(p^0,p^1,u,e_1,e_2,\ldots,e_H)$, pertaining to periods 0 and 1 for the arbitrary utility

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[144] This is the terminology used by Pollak (1989; 181) in his model of the conditional cost of living concept.
[145] Caves, Christensen and Diewert (1982; 1409) used the terms *demographic variables* or *public goods* to describe the vector of conditioning variables e in their generalized model of the Konüs price index or cost of living index.
[146] We assume that each $f^h(q,e)$ is continuous and increasing in the components of q and e and is concave in the components of q.
[147] In order to minimize notational clutter, in this section we use the notation $p\cdot q \equiv \sum_{n=1}^N p_nq_n$ as the inner product between the vectors p and q, rather than write out the summations.
[148] These authors provided generalizations of the plutocratic cost of living index due to Prais (1959). Pollak and Diewert did not include the environmental variables in their definitions of a group cost of living index.

vector of household utilities $u \equiv (u_1,u_2,\ldots,u_H)$ and for the arbitrary vectors of household environmental variables e_h for $h = 1,2,\ldots,H$ as follows:

$$(3.159)\quad P^*(p_1^0,\ldots,p_H^0,p_1^1,\ldots,p_H^1,u,e_1,e_2,\ldots,e_H) \equiv \sum_{h=1}^H C^h(u_h,e_h,p_h^1) / \sum_{h=1}^H C^h(u_h,e_h,p_h^0) .$$

The numerator on the right hand side of (3.159) is the sum over households of the minimum cost, $C^h(u_h,e_h,p_h^1)$, for household h to achieve the arbitrary utility level u_h, given that the household h faces the arbitrary vector of household h environmental variables e_h and also faces the period 1 vector of prices p_h^1. The denominator on the right hand side of (3.159) is the sum over households of the minimum cost, $C^h(u_h,e_h,p_h^0)$, for household h to achieve the *same* arbitrary utility level u_h, given that the household faces the *same* arbitrary vector of household h environmental variables e_h and also faces the period 0 vector of prices p_h^0. Thus in the numerator and denominator of (3.159), only the price variables are different, which is precisely what we want in a theoretical definition of a consumer price index.

173. We now specialize the general definition (3.159) by replacing the general utility vector u by either the period 0 vector of household utilities $u^0 \equiv (u_1^0,u_2^0,\ldots u_H^0)$ or the period 1 vector of household utilities $u^1 \equiv (u_1^1,u_2^1,\ldots u_H^1)$. We also specialize the general definition (3.159) by replacing the general household environmental vectors $(e_1,e_2,\ldots e_H) \equiv e$ by either the period 0 vector of household environmental variables $e^0 \equiv (e_1^0,e_2^0,\ldots e_H^0)$ or the period 1 vector of household environmental variables $e^1 \equiv (e_1^1,e_2^1,\ldots,e_H^1)$. The choice of the base period vector of utility levels and base period environmental variables leads to the *Laspeyres conditional plutocratic cost of living index*, $P^*(p_1^0,\ldots,p_H^0,p_1^1,\ldots,p_H^1,u^0,e^0)$[149], while the choice of the period 1 vector of utility levels and period 1 environmental variables leads to the *Paasche conditional plutocratic cost of living index*, $P^*(p_1^0,\ldots,p_H^0,p_1^1,\ldots,p_H^1,u^1,e^1)$. It turns out that these last two indices satisfy some interesting inequalities, which we derive below.

174. Using definition (3.159), the *Laspeyres plutocratic conditional cost of living index*, $P^*(p_1^0,\ldots,p_H^0,p_1^1,\ldots,p_H^1,u^0,e^0)$, may be written as follows:

$$\begin{aligned}(3.160)\quad & P^*(p_1^0,\ldots,p_H^0,p_1^1,\ldots,p_H^1,u^0,e_1^0,e_2^0,\ldots,e_H^0)\\
&\equiv \sum_{h=1}^H C^h(u_h^0,e_h^0,p_h^1) / \sum_{h=1}^H C^h(u_h^0,e_h^0,p_h^0)\\
&= \sum_{h=1}^H C^h(u_h^0,e_h^0,p_h^1) / \sum_{h=1}^H p_h^0\cdot q_h^0 \quad \text{using (3.158) for } t = 0\\
&\le \sum_{h=1}^H p_h^1\cdot q_h^0 / \sum_{h=1}^H p_h^0\cdot q_h^0\\
&\qquad \text{since } C^h(u_h^0,e_h^0,p_h^1) \equiv \min_q \{ p_h^1\cdot q : f^h(q,e_h^0) \ge u_h^0 \} \le p^1\cdot q_h^0 \text{ and } q_h^0\\
&\qquad \text{is feasible for the cost minimization problem for } h = 1,2,\ldots,H\\
&\equiv P_{PL}\end{aligned}$$

[149] This is the concept of a cost of living index that Triplett (2000; 27) found most useful for measuring inflation: "One might want to produce a COL *conditional* on the base period's weather experience…. In this case, the unusually cold winter does not affect the *conditional* COL subindex that holds the environment constant. … the COL subindex that holds the environment constant is probably the COL concept that is most useful for an anti-inflation policy." Hill (1999; 4) endorsed this point of view.

where P_{PL} is defined to be the observable (in principle) *plutocratic Laspeyres price index*, $\sum_{h=1}^{H} p_h^1 \cdot q_h^0 / \sum_{h=1}^{H} p_h^0 \cdot q_h^0$, which uses the individual vectors of household or regional quantities for period 0, $(q_1^0,\ldots,q_H^0)$, as quantity weights.[150] If prices are equal across households (or regions), so that

(3.161) $p_h^t = p^t$ for t = 0,1 and h = 1,2,…,H,

then the plutocratic (or disaggregated) Laspeyres price index P_{PL} collapses down to the usual *aggregate Laspeyres index*, P_L; i.e., if (3.161) holds, then P_{PL} in (3.160) becomes

(3.162)
$$\begin{aligned} P_{PL} &\equiv \textstyle\sum_{h=1}^{H} p_h^1 \cdot q_h^0 / \sum_{h=1}^{H} p_h^0 \cdot q_h^0 \\ &= p^1 \cdot \textstyle\sum_{h=1}^{H} q_h^0 / p^0 \cdot \sum_{h=1}^{H} q_h^0 \\ &= p^1 \cdot q^0 / p^0 \cdot q^0 \\ &\equiv P_L \end{aligned}$$

where the total quantity vector in period t is defined as

(3.163) $q^t \equiv \sum_{h=1}^{H} q_h^t$ for t = 0,1.

The inequality (3.160) says that the theoretical Laspeyres plutocratic conditional cost of living index, $P^*(p_1^0,\ldots,p_H^0,p_1^1,\ldots,p_H^1,u^0,e^0)$, is bounded from above by the observable (in principle) plutocratic or disaggregated Laspeyres price index P_{PL}. The special case of inequality (3.160) when the equal prices assumption (3.161) holds was first obtained by Pollak (1989; 182) for the case of one household with environmental variables and by Pollak (1980; 276) for the many household case but where the environmental variables are absent from the household utility and cost functions.

175. In a similar manner, specializing definition (3.159), *the Paasche conditional plutocratic cost of living index*, $P^*(p_1^0,\ldots,p_H^0,p_1^1,\ldots,p_H^1,u^1,e^1)$, may be written as follows:

(3.164)
$$\begin{aligned} &P^*(p_1^0,\ldots,p_H^0,p_1^1,\ldots,p_H^1,u^1,e_1^1,e_2^1,\ldots,e_H^1) \\ &\equiv \textstyle\sum_{h=1}^{H} C^h(u_h^1,e_h^1,p_h^1) / \sum_{h=1}^{H} C^h(u_h^1,e_h^1,p_h^0) \\ &= \textstyle\sum_{h=1}^{H} p_h^1 \cdot q_h^1 / \sum_{h=1}^{H} C^h(u_h^1,e_h^1,p_h^0) \qquad \text{using (3.158) for t = 1} \\ &\geq \textstyle\sum_{h=1}^{H} p_h^1 \cdot q_h^1 / \sum_{h=1}^{H} p_h^0 \cdot q_h^1 \qquad \text{using a feasibility argument} \\ &\equiv P_{PP} \end{aligned}$$

where P_{PP} is defined to be the *plutocratic or disaggregated (over households) Paasche price index*, $\sum_{h=1}^{H} p_h^1 \cdot q_h^1 / \sum_{h=1}^{H} p_h^0 \cdot q_h^1$, which uses the individual vectors of household quantities for period 1, $(q_1^1,\ldots,q_H^1)$, as quantity weights.

176. If prices are equal across households (or regions), so that assumptions (3.161) hold, then the disaggregated Paasche price index P_{PP} collapses down to the usual aggregate Paasche index, P_P; i.e., if (3.161) holds, then P_{PP} in (3.164) becomes

[150] Thus the plutocratic Laspeyres index can be regarded as an ordinary Laspeyres index except that each commodity in each region is regarded as a separate commodity.

(3.165) $P_{PP} \equiv \sum_{h=1}^{H} p_h^1 \cdot q_h^1 / \sum_{h=1}^{H} p_h^0 \cdot q_h^1$
$= p^1 \cdot \sum_{h=1}^{H} q_h^1 / p^0 \cdot \sum_{h=1}^{H} q_h^1$
$= p^1 \cdot q^1 / p^0 \cdot q^1$
$\equiv P_P$.

177. Returning to the inequality (3.164), we see that the theoretical Paasche conditional plutocratic cost of living index, $P^*(p_1^0,\ldots,p_H^0,p_1^1,\ldots,p_H^1,u^1,e^1)$, is bounded from below by the observable plutocratic or disaggregated Paasche price index P_{PP}. Diewert (1983a; 191) first obtained the inequality (3.164) for the case where the environmental variables are absent from the household utility and cost functions and prices are equal across households.

178. In the following subsection, we shall show how to obtain a theoretical plutocratic cost of living index that is bounded from above and below rather than the theoretical indices that just have the one sided bounds in (3.160) and (3.164).

I.2 The Fisher plutocratic price index

179. Using the inequalities (3.160) and (3.164) and the continuity properties of the conditional plutocratic cost of living $P^*(p_1^0,\ldots,p_H^0,p_1^1,\ldots,p_H^1,u,e)$ defined by (3.159), it is possible to modify the method of proof used by Konüs (1924) and Diewert (1983a; 191) and establish the following result:[151]

Under our assumptions, there exists a reference utility vector $u^* \equiv (u_1^*,u_2^*,\ldots,u_H^*)$ such that the household h reference utility level u_h^* lies between the household h period 0 and 1 utility levels, u_h^0 and u_h^1 respectively for h = 1,…,H, and there exist household environmental vectors $e_h^* \equiv (e_{h1}^*,e_{h2}^*,\ldots,e_{hM}^*)$ such that the household h reference mth environmental variable e_{hm}^* lies between the household h period 0 and 1 levels for the mth environmental variable, e_{hm}^0 and e_{hm}^1 respectively for m = 1,2,…,M and h = 1,…,H, and the conditional plutocratic cost of living index $P^*(p_1^0,\ldots,p_H^0,p_1^1,\ldots,p_H^1,u^*,e^*)$ evaluated at this intermediate reference utility vector u^* and the intermediate reference vector of household environmental variables $e^* \equiv (e_1^*,e_2^*,\ldots,e_H^*)$ lies between the observable (in principle) plutocratic Laspeyres and Paasche price indices, P_{PL} and P_{PP}, defined above by the last equalities in (3.160) and (3.164).

180. The above result tells us that *the theoretical national plutocratic conditional consumer price index* $P^*(p_1^0,\ldots,p_H^0,p_1^1,\ldots,p_H^1,u^*,e^*)$ lies between the plutocratic or disaggregated Laspeyres index P_{PL} and the plutocratic or disaggregated Paasche index P_{PP}. Hence if P_{PL} and P_{PP} are not too different, a good point approximation to the theoretical national plutocratic consumer price index will be the *plutocratic or disaggregated Fisher index* P_{PF} defined as:

[151] Note that the household cost functions must be continuous in the environmental variables which is a real restriction on the types of environmental variables which can be accommodated by the result.

$$(3.166)\quad P_{PF} \equiv [P_{PL}\, P_{PP}]^{1/2} .$$

The plutocratic Fisher price index P_{PF} is computed just like the usual Fisher price index, except that each commodity in each region (or for each household) is regarded as a separate commodity. Of course, this index will satisfy the time reversal test.

181. Since statistical agencies do not calculate Laspeyres, Paasche and Fisher price indices by taking inner products of price and quantity vectors as we have done in (3.166), it will be useful to obtain formulae for the Laspeyres and Paasche indices that depend only on price relatives and expenditure shares. In order to do this, we need to introduce some notation. Define the *expenditure share of household h on commodity i in period t* as

$$(3.167)\quad s_{hi}^{t} \equiv p_{hi}^{t}\, q_{hi}^{t} / \textstyle\sum_{k=1}^{n} p_{hk}^{t}\, q_{hk}^{t} ; \qquad t = 0,1 ;\; h = 1,2,\ldots,H ;\; i = 1,2,\ldots,n.$$

Define the *expenditure share of household h in total period t consumption* as:

$$\begin{aligned}(3.168)\quad S_{h}^{t} &\equiv \textstyle\sum_{i=1}^{n} p_{hi}^{t}\, q_{hi}^{t} / \sum_{k=1}^{H} \sum_{i=1}^{n} p_{ik}^{t}\, q_{ik}^{t} \\ &= p_{h}^{t}\cdot q_{h}^{t} / \textstyle\sum_{k=1}^{H} p_{k}^{t}\cdot q_{k}^{t} \qquad t = 0,1 ;\; h = 1,2,\ldots,H.\end{aligned}$$

Finally, define the *national expenditure share of commodity i in period t* as:

$$\begin{aligned}(3.169)\quad \sigma_{i}^{t} &\equiv \textstyle\sum_{h=1}^{H} p_{hi}^{t}\, q_{hi}^{t} / \sum_{k=1}^{H} p_{k}^{t}\cdot q_{k}^{t} && t = 0,1 ;\; i = 1,2,\ldots,n \\ &= \textstyle\sum_{h=1}^{H} [p_{hi}^{t}\, q_{hi}^{t} / p_{h}^{t}\cdot q_{h}^{t}][\, p_{h}^{t}\cdot q_{h}^{t} / \sum_{k=1}^{H} p_{k}^{t}\cdot q_{k}^{t}] \\ &= \textstyle\sum_{h=1}^{H} s_{hi}^{t}\;\; p_{h}^{t}\cdot q_{h}^{t} / \sum_{k=1}^{H} p_{k}^{t}\cdot q_{k}^{t} && \text{using definitions (3.167)} \\ &= \textstyle\sum_{h=1}^{H} s_{hi}^{t}\, S_{h}^{t} && \text{using definitions (3.168).}\end{aligned}$$

The *Laspeyres price index for region h* (or household h) is defined as:

$$\begin{aligned}(3.170)\quad P_{Lh} &\equiv p_{h}^{1}\cdot q_{h}^{0} / \; p_{h}^{0}\cdot q_{h}^{0} && h = 1,2,\ldots,H \\ &= \textstyle\sum_{i=1}^{n} (p_{hi}^{1}/ p_{hi}^{0})\, p_{hi}^{0}\, q_{hi}^{0} / p_{h}^{0}\cdot q_{h}^{0} \\ &= \textstyle\sum_{i=1}^{n}\; s_{hi}^{0}(p_{hi}^{1}/ p_{hi}^{0}) && \text{using definitions (3.167).}\end{aligned}$$

182. Referring back to (3.160), the *plutocratic national Laspeyres price index* P_{PL} can be rewritten as follows:

$$\begin{aligned}(3.171)\quad P_{PL} &\equiv \textstyle\sum_{h=1}^{H}\; p_{h}^{1}\cdot q_{h}^{0} / \sum_{h=1}^{H} p_{h}^{0}\cdot q_{h}^{0} \\ &= \textstyle\sum_{h=1}^{H}\; [p_{h}^{1}\cdot q_{h}^{0} / p_{h}^{0}\cdot q_{h}^{0}][\, p_{h}^{0}\cdot q_{h}^{0} / \sum_{h=1}^{H} p_{h}^{0}\cdot q_{h}^{0}] \\ &= \textstyle\sum_{h=1}^{H}\; [p_{h}^{1}\cdot q_{h}^{0} / p_{h}^{0}\cdot q_{h}^{0}]\, S_{h}^{0} && \text{using definitions (3.168) with } t = 0 \\ (3.172)\quad &= \textstyle\sum_{h=1}^{H} S_{h}^{0}\, P_{Lh} && \text{using definitions (3.170)} \\ &= \textstyle\sum_{h=1}^{H} S_{h}^{0} \sum_{i=1}^{n}\; s_{hi}^{0}(p_{hi}^{1}/ p_{hi}^{0}) && \text{using the last line of (3.170)} \\ (3.173)\quad &= \textstyle\sum_{h=1}^{H} \sum_{i=1}^{n} S_{h}^{0}\, s_{hi}^{0}(p_{hi}^{1}/ p_{hi}^{0}) && \text{rearranging terms.}\end{aligned}$$

Equation (3.172) shows that the plutocratic national Laspeyres price index is equal to a (period 0) regional expenditure share weighted average of the regional Laspeyres price indices. Equation (3.173) shows that the national Laspeyres price index is equal to a period 0 expenditure share weighted average of the regional price relatives, (p_{hi}^1/ p_{hi}^0), where the corresponding weight, $S_h^0 s_{hi}^0$, is the period 0 national expenditure share of commodity i in region h.

183. The *Paasche price index for region h* (or household h) is defined as:

$$\begin{aligned}(3.174)\quad P_{Ph} &\equiv p_h^1 \cdot q_h^1 / p_h^0 \cdot q_h^1 && h = 1,2,\ldots,H \\ &= 1 / \textstyle\sum_{i=1}^n (p_{hi}^0/ p_{hi}^1)\, p_{hi}^1\, q_{hi}^1 / p_h^1 \cdot q_h^1 \\ &= 1 / \textstyle\sum_{i=1}^n s_{hi}^1 (p_{hi}^1/ p_{hi}^0)^{-1} && \text{using definitions (3.167)} \\ &= [\textstyle\sum_{i=1}^n s_{hi}^1 (p_{hi}^1/ p_{hi}^0)^{-1}]^{-1} .\end{aligned}$$

184. Referring back to (3.165), the *plutocratic national Paasche price index* P_{PP} can be rewritten as follows:

$$\begin{aligned}(3.175)\quad P_{PP} &\equiv \textstyle\sum_{k=1}^H p_k^1 \cdot q_k^1 / \sum_{h=1}^H p_h^0 \cdot q_h^1 \\ &= 1 / \{\textstyle\sum_{h=1}^H [p_h^0 \cdot q_h^1 / p_h^1 \cdot q_h^1][p_h^1 \cdot q_h^1 / \sum_{k=1}^H p_k^1 \cdot q_k^1]\} \\ &= 1 / \textstyle\sum_{h=1}^H [p_h^1 \cdot q_h^0 / p_h^0 \cdot q_h^0]^{-1} S_h^1 && \text{using definitions (3.168) with t = 1} \\ (3.176)\quad &= [\textstyle\sum_{h=1}^H S_h^1 P_{Ph}^{-1}]^{-1} && \text{using definitions (3.174)} \\ &= [\textstyle\sum_{h=1}^H S_h^1 \sum_{i=1}^n s_{hi}^1 (p_{hi}^1/ p_{hi}^0)^{-1}]^{-1} && \text{using the last line of (3.174)} \\ (3.177)\quad &= [\textstyle\sum_{h=1}^H \sum_{i=1}^n S_h^1 s_{hi}^1 (p_{hi}^1/ p_{hi}^0)^{-1}]^{-1} && \text{rearranging terms.}\end{aligned}$$

Equation (3.176) shows that the national plutocratic Paasche price index is equal to a (period 1) regional expenditure share *weighted harmonic mean* of the regional Paasche price indices. Equation (3.177) shows that the national Paasche price index is equal to a period 1 expenditure share *weighted harmonic average* of the regional price relatives, (p_{hi}^1/ p_{hi}^0), where the weight for this price relative, $S_h^1 s_{hi}^1$, is the period 1 national expenditure share of commodity i in region h.

185. Of course, the share formulae for the plutocratic Paasche and Laspeyres indices, P_{PP} and P_{PL}, given by (3.177) and (3.173) can now be used to calculate the plutocratic Fisher index, $P_{PF} \equiv [P_{PP}\, P_{PL}]^{1/2}$.

186. If prices are equal across regions, the formulae (3.173) and (3.177) simplify. The formula for the plutocratic Laspeyres index (3.173) becomes:

$$\begin{aligned}(3.178)\quad P_{PL} &= \textstyle\sum_{h=1}^H \sum_{i=1}^n S_h^0 s_{hi}^0 (p_{hi}^1/ p_{hi}^0) \\ &= \textstyle\sum_{h=1}^H \sum_{i=1}^n S_h^0 s_{hi}^0 (p_i^1/ p_i^0) && \text{using assumptions (3.161)} \\ &= \textstyle\sum_{i=1}^n \sigma_i^0 (p_i^1/ p_i^0) && \text{using (3.169) for t = 0} \\ &= P_L\end{aligned}$$

where P_L is the usual aggregate Laspeyres price index based on the assumption that each household faces the same vector of commodity prices; see (3.162) for the definition of P_L. Under the equal prices across households assumption (3.161), the formula for the plutocratic Paasche index (3.177) becomes:

$$(3.179)\ P_{PP} = [\sum_{h=1}^{H} \sum_{i=1}^{n} S_h^{1} s_{hi}^{1}(p_{hi}^{1}/ p_{hi}^{0})^{-1}]^{-1}$$
$$= [\sum_{h=1}^{H} \sum_{i=1}^{n} S_h^{1} s_{hi}^{1}(p_i^{1}/ p_i^{0})^{-1}]^{-1} \qquad \text{using assumptions (3.161)}$$
$$= [\sum_{i=1}^{n} \sigma_i^{1} (p_i^{1}/ p_i^{0})^{-1}]^{-1} \qquad \text{using (3.169) for } t = 1$$
$$= P_P$$

where P_P is the usual aggregate Paasche price index based on the assumption that each household faces the same vector of commodity prices; see (3.165) for the definition of P_P.

187. Thus with the assumption that commodity prices are the same across regions, in order to calculate national Laspeyres and Paasche indices, we require only "national" price relatives and national commodity expenditure shares for the two periods under consideration. However, if there is regional variation in prices, then the simplified formulae (3.178) and (3.179) are not valid and we must use our earlier formulae, (3.173) and (3.177).

I.3 Democratic versus plutocratic cost of living indices

188. The plutocratic indices considered above weight each household in the economy according to the size of their expenditures in the two periods under consideration. Instead of weighting in this way, it is possible to define theoretical indices (and "practical" approximations to them) that give each household or household group in the economy an *equal weight*. Following Prais (1959), we will call such an index a *democratic index*. In this subsection, we will rework the plutocratic index number theory developed in subsections I.1 and I.2 above into a democratic framework.

189. Making the same assumptions as in section I.1 above, we define the class of *conditional democratic cost of living indices*, $P_D^*(p^0,p^1,u,e_1,e_2,\ldots,e_H)$, pertaining to periods 0 and 1 for the arbitrary utility vector of household utilities $u \equiv (u_1,u_2,\ldots,u_H)$ and for the arbitrary vectors of household environmental variables e_h for $h = 1,2,\ldots,H$ as follows:

$$(3.180)\quad P_D^*(p_1^0,\ldots,p_H^0,p_1^1,\ldots,p_H^1,u,e_1,e_2,\ldots,e_H) \equiv \sum_{h=1}^{H} [1/H]C^h(u_h,e_h,p_h^1)/C^h(u_h,e_h,p_h^0).$$

Thus P_D^* is a simple unweighted arithmetic average of the individual household *conditional cost of living indices*, $C^h(u_h,e_h,p_h^1)/C^h(u_h,e_h,p_h^0)$. In the numerator and denominator of these conditional indices, only the price variables are different, which is precisely what we want in a theoretical definition of a consumer price index. If the vector of environmental variables, e_h, is not present in the cost function of household h, then the conditional index $C^h(u_h,e_h,p_h^1)/C^h(u_h,e_h,p_h^0)$ becomes an ordinary Konüs true cost of living index of the type defined earlier by (3.81).

190. We now specialize the general definition (3.180) by replacing the general utility vector u by either the period 0 vector of household utilities $u^0 \equiv (u_1^0,u_2^0,\ldots u_H^0)$ or the period 1 vector of household utilities $u^1 \equiv (u_1^1,u_2^1,\ldots u_H^1)$. We also specialize the general definition (3.180) by replacing the general household environmental vectors $(e_1,e_2,\ldots e_H)$

$\equiv e$ by either the period 0 vector of household environmental variables $e^0 \equiv (e_1^0,e_2^0,\ldots e_H^0)$ or the period 1 vector of household environmental variables $e^1 \equiv (e_1^1,e_2^1,\ldots,e_H^1)$. The choice of the base period vector of utility levels and base period environmental variables leads to the *Laspeyres conditional democratic cost of living index*, $P_D^*(p_1^0,\ldots,p_H^0,p_1^1,\ldots,p_H^1,u^0,e^0)$, while the choice of the period 1 vector of utility levels and period 1 environmental variables leads to the *Paasche conditional democratic cost of living index*, $P_D^*(p_1^0,\ldots,p_H^0,p_1^1,\ldots,p_H^1,u^1,e^1)$. It turns out that these two democratic indices satisfy some interesting inequalities, which we derive below.

191. Specializing definition (3.180), the *Laspeyres democratic conditional cost of living index*, $P_D^*(p_1^0,\ldots,p_H^0,p_1^1,\ldots,p_H^1,u^0,e^0)$, may be written as follows:

$$\begin{aligned}
(3.181)\quad & P_D^*(p_1^0,\ldots,p_H^0,p_1^1,\ldots,p_H^1,u^0,e_1^0,e_2^0,\ldots,e_H^0) \\
& \equiv \textstyle\sum_{h=1}^H [1/H]\, C^h(u_h^0,e_h^0,p_h^1) / C^h(u_h^0,e_h^0,p_h^0) \\
& = \textstyle\sum_{h=1}^H [1/H]\, C^h(u_h^0,e_h^0,p_h^1) / p_h^0\cdot q_h^0 \qquad \text{using (3.158) for } t = 0 \\
& \leq \textstyle\sum_{h=1}^H [1/H]\, p_h^1\cdot q_h^0 / p_h^0\cdot q_h^0 \\
& \qquad \text{since } C^h(u_h^0,e_h^0,p_h^1) \equiv \min_q \{ p_h^1\cdot q : f^h(q,e_h^0) \geq u_h^0 \} \leq p^1\cdot q_h^0 \text{ and } q_h^0 \\
& \qquad \text{is feasible for the cost minimization problem for } h = 1,2,\ldots,H \\
& \equiv P_{DL}
\end{aligned}$$

where P_{DL} is defined to be the observable (in principle) *democratic Laspeyres price index*, $\sum_{h=1}^H [1/H]\, p_h^1\cdot q_h^0 / p_h^0\cdot q_h^0$, which uses the individual vectors of household or regional quantities for period 0, $(q_1^0,\ldots,q_H^0)$, as quantity weights.

192. In a similar manner, specializing definition (3.180), *the Paasche conditional democratic cost of living index*, $P_D^*(p_1^0,\ldots,p_H^0,p_1^1,\ldots,p_H^1,u^1,e^1)$, may be written as follows:

$$\begin{aligned}
(3.182)\quad & P_D^*(p_1^0,\ldots,p_H^0,p_1^1,\ldots,p_H^1,u^1,e_1^1,e_2^1,\ldots,e_H^1) \\
& \equiv \textstyle\sum_{h=1}^H [1/H]\, C^h(u_h^1,e_h^1,p_h^1) / C^h(u_h^1,e_h^1,p_h^0) \\
& = \textstyle\sum_{h=1}^H [1/H]\, p_h^1\cdot q_h^1 / C^h(u_h^1,e_h^1,p_h^0) \qquad \text{using (3.158) for } t = 1 \\
& \geq \textstyle\sum_{h=1}^H [1/H]\, p_h^1\cdot q_h^1 / p_h^0\cdot q_h^1 \qquad \text{using a feasibility argument} \\
& \equiv P_{DP}
\end{aligned}$$

where P_{DP} is defined to be the *democratic Paasche price index*, $\sum_{h=1}^H [1/H]\, p_h^1\cdot q_h^1 / p_h^0\cdot q_h^1$, which uses the individual vector of household h quantities for period 1, q_h^1, as quantity weights for term h in the summation of individual household Paasche indices. Thus, we see that the theoretical Paasche conditional democratic cost of living index, $P_D^*(p_1^0,\ldots,p_H^0,p_1^1,\ldots,p_H^1,u^1,e^1)$, is bounded from below by the observable (in principle) democratic Paasche price index P_{DP}. Diewert (1983a; 191) first obtained the inequality (3.182) for the case where the environmental variables are absent from the household utility and cost functions and prices are equal across households.

193. We now show how to obtain a theoretical democratic cost of living index that is bounded from above and below by observable indices. Using the inequalities (3.181) and

(3.182) and the continuity properties of the conditional democratic cost of living $P^*(p_1^0,\ldots,p_H^0,p_1^1,\ldots,p_H^1,u,e)$ defined by (3.180), it is possible to modify the method of proof used by Konüs (1924) and Diewert (1983a; 191) and establish the following result:

Under our assumptions, there exists a reference utility vector $u^* \equiv (u_1^*,u_2^*,\ldots,u_H^*)$ such that the household h reference utility level u_h^* lies between the household h period 0 and 1 utility levels, u_h^0 and u_h^1 respectively for h = 1,…,H, and there exist household environmental vectors $e_h^* \equiv (e_{h1}^*,e_{h2}^*,\ldots,e_{hM}^*)$ such that the household h reference mth environmental variable e_{hm}^* lies between the household h period 0 and 1 levels for the mth environmental variable, e_{hm}^0 and e_{hm}^1 respectively for m = 1,2,…,M and h = 1,…,H, and the conditional democratic cost of living index $P_D^*(p_1^0,\ldots,p_H^0,p_1^1,\ldots,p_H^1,u^*,e^*)$ evaluated at this intermediate reference utility vector u^* and the intermediate reference vector of household environmental variables $e^* \equiv (e_1^*,e_2^*,\ldots,e_H^*)$ lies between the observable (in principle) democratic Laspeyres and Paasche price indices, P_{DL} and P_{DP}, defined above by the last equalities in (3.181) and (3.182).

194. The above result tells us that *the theoretical national democratic conditional consumer price index* $P_D^*(p_1^0,\ldots,p_H^0,p_1^1,\ldots,p_H^1,u^*,e^*)$ lies between the democratic Laspeyres index P_{DL} and the democratic Paasche index P_{DP}. Hence if P_{DL} and P_{DP} are not too different, a good point approximation to the theoretical national democratic consumer price index will be the *democratic Fisher index* P_{DF} defined as:

$$(3.183)\quad P_{DF} \equiv [P_{DL}\, P_{DP}]^{1/2}\,.$$

The democratic Fisher price index P_{DF} will satisfy the time reversal test.

195. Again, it will be useful to obtain formulae for the democratic Laspeyres and Paasche indices that depend only on price relatives and expenditure shares. Using definition (3.167) for the household h expenditure share on commodity i during period t, s_{hi}^t, the Laspeyres and Paasche price indices for household h can be written in share form as follows:

$$(3.184)\quad P_{Lh} \equiv p_h^1\cdot q_h^0 /\ p_h^0\cdot q_h^0 = \sum_{i=1}^n s_{hi}^0\,(p_{hi}^1/p_{hi}^0)\,; \qquad h = 1,\ldots,H\,;$$

$$(3.185)\quad P_{Ph} \equiv p_h^1\cdot q_h^1 /\ p_h^0\cdot q_h^1 = [\sum_{i=1}^n s_{hi}^1\,(p_{hi}^1/p_{hi}^0)^{-1}]^{-1}\,; \qquad h = 1,\ldots,H\,.$$

Substituting (3.184) into the definition of the democratic Laspeyres index, P_{DL}, leads to the following share type formula:[152]

$$(3.186)\quad P_{DL} = \sum_{h=1}^H [1/H] \sum_{i=1}^n s_{hi}^0\,(p_{hi}^1/p_{hi}^0).$$

[152] Comparing the formula for the democratic Laspeyres index, P_{DL}, with the previous formula (3.173) for the plutocratic Laspeyres index, P_{PL}, we see that the plutocratic weight for the ith price relative for household h is $S_h^0 s_{hi}^0$ whereas the corresponding democratic weight is $(1/H)s_{hi}^0$. Thus households that have larger base period expenditures and hence bigger expenditure shares S_h^0 get a larger weight in the plutocratic index as compared to the democratic index.

Similarly, substituting (3.185) into the definition of the democratic Paasche index, P_{DP}, leads to the following share type formula:

(3.187) $P_{DP} = \sum_{h=1}^{H} [1/H] [\sum_{i=1}^{n} s_{hi}^{1} (p_{hi}^{1}/p_{hi}^{0})^{-1}]^{-1}$.

196. The formula for the democratic Laspeyres index in the previous paragraph simplifies if we can assume that each household faces the same vector of prices in each of the two periods under consideration. Under this condition, we may rewrite (3.186) as

(3.188) $P_{DL} = \sum_{i=1}^{n} s_{di}^{0} (p_{i}^{1}/p_{i}^{0})$

where the *period 0 democratic expenditure share for commodity i*, s_{di}^{0}, is defined as follows:

(3.189) $s_{di}^{0} \equiv \sum_{h=1}^{H} [1/H] s_{hi}^{0}$; $i = 1,\ldots,n$.

Thus s_{di}^{0} is simply the arithmetic average (over all households) of the individual household expenditure shares on commodity i during period 0. The formula for the democratic Paasche index does not simplify in the same way, under the assumption that households face the same prices in each period, due to the harmonic form of averaging in (3.185).

197. Our conclusion at this point is that democratic and plutocratic Laspeyres, Paasche and Fisher indices can be constructed by a statistical agency provided that information on household specific price relatives p_{hi}^{1}/p_{hi}^{0} and expenditures is available for both periods under consideration. If expenditure information is available only for the base period, then only the Laspeyres democratic and plutocratic indices can be constructed.

198. It is now necessary to discuss a practical problem that faces statistical agencies: namely, that existing household consumer expenditure surveys, which are used in order to form estimates of household expenditure shares, *are not very accurate*. Thus the detailed commodity by region expenditure shares, $S_h^{0}s_{hn}^{0}$ and $S_h^{1}s_{hn}^{1}$, which appear in the formulae for the plutocratic Laspeyres and Paasche indices are generally measured with very large errors. Similarly, the individual household expenditure shares for the two periods under consideration, s_{hn}^{0} and s_{hn}^{1}, which are required in order to calculate the democratic Laspeyres and Paasche indices defined by (3.186) and (3.187) respectively, are also generally measured with substantial errors. Hence, it may lead to less overall error if the regional commodity expenditure shares s_{hn}^{t} are replaced by the national commodity expenditure shares σ_n^{t} defined by (3.169). Whether this approximation is justified would depend on a detailed analysis of the situation facing the statistical agency. In general, complete and accurate information on household expenditure shares will not be available to the statistical agency and hence statistical estimation and smoothing techniques will have to be used in order to obtain expenditure weights that will be used to weight the price relatives collected by the agency.

199. This completes our introduction to the economic approach to index number theory. Other aspects of the economic approach will be covered in subsequent chapters.[153]

Appendix 3.1 The relationship between the Paasche and Laspeyres indices

1. Recall the notation used in section B.2 above. Define the ith relative price or price relative r_i and the ith quantity relative t_i as follows:

$$(A3.1.1)\quad r_i \equiv p_i^1/p_i^0 \ ; \ t_i \equiv q_i^1/q_i^0 \ ; \ i = 1,\ldots,n.$$

Using formula (3.8) above for the Laspeyres price index P_L and definitions (A3.1.1), we have:

$$(A3.1.2)\quad P_L = \sum_{i=1}^n r_i s_i^0 \equiv r^* ;$$

i.e., we define the “average” price relative r* as the base period expenditure share weighted average of the individual price relatives, r_i .

2. Using formula (3.6) for the Paasche price index P_P, we have:

$$\begin{aligned}(A3.1.3)\quad P_P &\equiv \sum_{i=1}^n p_i^1 q_i^1 / \sum_{m=1}^n p_m^0 q_m^1 \\ &= \sum_{i=1}^n r_i t_i p_i^0 q_i^0 / \sum_{m=1}^n t_m p_m^0 q_m^0 \quad \text{using definitions (A3.1.1)} \\ &= \sum_{i=1}^n r_i t_i s_i^0 / \sum_{m=1}^n t_m s_m^0 \\ &\qquad \text{dividing numerator and denominator by } \sum_{i=1}^n p_i^0 q_i^0 \\ &= \{[1/\sum_{m=1}^n t_m s_m^0][\sum_{i=1}^n (r_i - r^*)(t_i - t^*) s_i^0]\} + r^*\end{aligned}$$

using (A3.1.2) and $\sum_{i=1}^n s_i^0 = 1$ and where the “average” quantity relative t* is defined as

$$\begin{aligned}(A3.1.4)\quad t^* &\equiv \sum_{i=1}^n t_i s_i^0 \\ &= Q_L\end{aligned}$$

where the last equality follows using (3.11), the definition of the Laspeyres quantity index Q_L.

3. Taking the difference between P_P and P_L and using (A3.1.2)-(A3.1.4) yields:

$$(A3.1.5)\quad P_P - P_L = [1/Q_L][\sum_{i=1}^n (r_i - r^*)(t_i - t^*) s_i^0].$$

Now let r and t be discrete random variables that take on the n values r_i and t_i respectively. Let s_i^0 be the joint probability that $r = r_i$ and $t = t_i$ for i = 1,…,n and let the joint probability be 0 if $r = r_i$ and $t = t_j$ where $i \neq j$. It can be verified that the summation $\sum_{i=1}^n (r_i - r^*)(t_i - t^*) s_i^0$ on the right hand side of (A3.1.5) is the covariance between the

[153] For criticisms and limitations of the economic approach, see Turvey (2000) and Diewert (2001). For a vigorous defense of the economic approach, see Triplett (2000).

price relatives r_i and the corresponding quantity relatives t_i. This covariance can be converted into a correlation coefficient.[154] If this covariance is negative, which is the usual case in the consumer context, then P_P will be less than P_L.

Appendix 3.2 The relationship between the Divisia and economic approaches

1. Divisia's approach to index number theory relied on the theory of differentiation. Thus it does not appear to have any connection with economic theory. However, starting with Ville (1946), a number of economists[155] have established that the Divisia price and quantity indices *do* have a connection with the economic approach to index number theory. We outline this connection.

2. We first outline the economic approach to the determination of the price level and the quantity level. The particular economic approach that we use in this appendix is due to Shephard (1953) (1970) and Samuelson and Swamy (1974). This homothetic preferences approach was explained in section H.2 of the main text for this chapter.

3. As in section H.1, we assume that "the" consumer has well defined *preferences* over different combinations of the n consumer commodities or items. Each combination of items can be represented by a positive vector $q \equiv [q_1,\ldots,q_n]$. The consumer's preferences over alternative possible consumption vectors q are assumed to be representable by a continuous, nondecreasing and concave utility function f. We further assume that the consumer minimizes the cost of achieving the period t utility level $u^t \equiv f(q^t)$ for periods t = 0,1,…,T. Thus we assume that the observed period t consumption vector q^t solves the following period t cost minimization problem:

$$\text{(A3.2.1)}\quad C(u^t,p^t) \equiv \min_q \{\textstyle\sum_{i=1}^n p_i^t q_i : f(q) = u^t = f(q^t)\} = \textstyle\sum_{i=1}^n p_i^t q_i^t ; \quad t = 0,1,\ldots,T.$$

The period t price vector for the n commodities under consideration that the consumer faces is p^t. Note that the solution to the period t cost or expenditure minimization problem defines the *consumer's cost function*, $C(u^t,p^t)$.

4. As in section H.2, we place an additional regularity condition on the consumer's utility function f. We assume that f is (positively) linearly homogeneous for strictly positive quantity vectors. Under this assumption, the consumer's expenditure or cost function, C(u,p), decomposes into uc(p) where c(p) is the consumer's unit cost function; see equation (3.85) in section H.2 above. Under these assumptions, we obtain the following counterparts to equations (3.86) in section H.2:

$$\text{(A3.2.2)}\quad \textstyle\sum_{i=1}^n p_i^t q_i^t = c(p^t)f(q^t) \qquad \text{for } t = 0,1,\ldots,T.$$

[154] See Bortkiewicz (1923; 374-375) for the first application of this correlation coefficient decomposition technique.

[155] See for example Malmquist (1953; 227), Wold (1953; 134-147), Solow (1957), Jorgenson and Griliches (1967) and Hulten (1973). See Balk (2000a) for a recent survey of work on Divisia price and quantity indices.

Thus the period t total expenditure on the n commodities in the aggregate, $\sum_{i=1}^{n} p_i^t q_i^t$, decomposes into the product of two terms, $c(p^t)f(q^t)$. We can identify the period t unit cost, $c(p^t)$, as the period t price level P^t and the period t level of utility, $f(q^t)$, as the period t quantity level Q^t.

5. We now relate the economic price level for period t, $P^t \equiv c(p^t)$, that was defined in the previous paragraph to the Divisia price level for time t, P(t), that was implicitly defined by the differential equation (3.29). As in section D.1 above, we now think of the prices as being continuous, differentiable functions of time, $p_i(t)$ say, for i = 1,…,n. Thus the unit cost function can be regarded as a function of time t as well; i.e., we define the unit cost function as a function of t as

(A3.2.3) $c^*(t) \equiv c[p_1(t),p_2(t),\ldots,p_n(t)]$.

Assuming that the first order partial derivatives of the unit cost function c exist, we can calculate the logarithmic derivative of c*(t) as follows:

(A3.2.4) $$\begin{aligned} d\ln c^*(t)/dt &\equiv [1/c^*(t)]\, dc^*(t)/dt \\ &= [1/c^*(t)] \sum_{i=1}^{n} c_i[p_1(t),p_2(t),\ldots,p_n(t)]\, p_i'(t) && \text{using (A3.2.3)} \end{aligned}$$

where $c_i[p_1(t),p_2(t),\ldots,p_N(t)] \equiv \partial c[p_1(t),p_2(t),\ldots,p_n(t)]/\partial p_i$ is the partial derivative of the unit cost function with respect to the ith price, p_i, and $p_i'(t) \equiv dp_i(t)/dt$ is the time derivative of the ith price function, $p_i(t)$. Using Shephard’s (1953; 11) lemma, the consumer’s cost minimizing demand for commodity i at time t is:

(A3.2.5) $q_i(t) = u(t)\, c_i[p_1(t),p_2(t),\ldots,p_n(t)]$ for i = 1,…,n

where the utility level at time t is $u(t) = f[q_1(t),q_2(t),\ldots,q_n(t)]$. The continuous time counterpart to equations (A3.2.2) above is that total expenditure at time t is equal to total cost at time t which in turn is equal to the utility level, u(t), times the period t unit cost, c*(t); i.e., we have:

(A3.2.6) $\sum_{i=1}^{n} p_i(t)q_i(t) = u(t)c^*(t) = u(t)\, c[p_1(t),p_2(t),\ldots,p_n(t)]$.

Now the logarithmic derivative of the Divisia price level P(t) can be written as (recall (3.29) above):

(A3.2.7) $$\begin{aligned} P'(t)/P(t) &= \sum_{i=1}^{n} p_i'(t)q_i(t) \,/\, \sum_{i=1}^{n} p_i(t)q_i(t) \\ &= \sum_{i=1}^{n} p_i'(t)q_i(t) \,/\, u(t)\, c^*(t) && \text{using (A3.2.6)} \\ &= \sum_{i=1}^{n} p_i'(t)\{u(t)\, c_i[p_1(t),p_2(t),\ldots,p_n(t)]\} / u(t)\, c^*(t) && \text{using (A3.2.5)} \\ &= \sum_{i=1}^{n} c_i[p_1(t),p_2(t),\ldots,p_n(t)]\, p_i'(t) / c^*(t) && \text{rearranging terms} \\ &= [1/c^*(t)]\, dc^*(t)/dt && \text{using (A3.2.4)} \\ &\equiv c^{*\prime}(t)/c^*(t). \end{aligned}$$

Thus under the above continuous time cost minimizing assumptions, the Divisia price level, P(t), is essentially equal to the unit cost function evaluated at the time t prices, c*(t) ≡ $c[p_1(t),p_2(t),\ldots,p_N(t)]$.

6. If the Divisia price level P(t) is set equal to the unit cost function c*(t) ≡ $c[p_1(t),p_2(t),\ldots,p_N(t)]$, then from (A3.2.2), it follows that the Divisia quantity level Q(t) defined by (3.30) will equal the consumer's utility function regarded as a function of time, f*(t) ≡ $f[q_1(t),\ldots,q_n(t)]$. Thus under the assumption that the consumer is continuously minimizing the cost of achieving a given utility level where the utility or preference function is linearly homogeneous, we have shown that the Divisia price and quantity levels P(t) and Q(t), defined implicitly by the differential equations (3.29) and (3.30), are essentially equal to the consumer's unit cost function c*(t) and utility function f*(t) respectively.[156] These are rather remarkable equalities since in principle, given the functions of time, $p_i(t)$ and $q_i(t)$, we can solve the differential equations numerically and hence P(t) and Q(t) are in principle observable (up to some normalizing constants).

7. For more on the Divisia approach to index number theory, see Vogt (1977)(1978) and Balk (2000a). An alternative approach to Divisia indices using line integrals may be found in the companion volume on the Producer Price Index; see Eurostat, ILO, IMF, OECD, UNECE and the World Bank (2000).

Appendix 3.3 Price Indices using an Artificial Data Set

1. In order to give the reader some idea of how much the various index numbers defined in this chapter might differ using a "real" data set, we compute all of the major indices defined in the chapter using an artificial data set consisting of prices and quantities for 6 commodities over 5 periods. The period can be thought of as somewhere between a year and 5 years. The trends in the data are generally more pronounced than one would see in the course of a year. The price and quantity data are listed in Tables 3.3.1 and 3.3.2 below. For convenience, we have also listed the period t nominal expenditures, $p^t \cdot q^t \equiv \sum_{i=1}^n p_i^t q_i^t$, along with the corresponding period t expenditure shares, $s_i^t \equiv p_i^t q_i^t / p^t \cdot q^t$, in Table 3.3.3.

Table 3.3.1 Prices for Six Commodities

Period t	p_1^t	p_2^t	p_3^t	p_4^t	p_5^t	p_6^t
1	1.0	1.0	1.0	1.0	1.0	1.0
2	1.2	3.0	1.3	0.7	1.4	0.8
3	1.0	1.0	1.5	0.5	1.7	0.6
4	0.8	0.5	1.6	0.3	1.9	0.4
5	1.0	1.0	1.6	0.1	2.0	0.2

Table 3.3.2 Quantities for Six Commodities

[156] Obviously, the scale of the utility and cost functions are not uniquely determined by the differential equations (3.29) and (3.30).

Period t	q_1^t	q_2^t	q_3^t	q_4^t	q_5^t	q_6^t
1	1.0	1.0	2.0	1.0	4.5	0.5
2	0.8	0.9	1.9	1.3	4.7	0.6
3	1.0	1.1	1.8	3.0	5.0	0.8
4	1.2	1.2	1.9	6.0	5.6	1.3
5	0.9	1.2	2.0	12.0	6.5	2.5

Table 3.3.3 Expenditures and Expenditure Shares for Six Commodities

Period t	$p^t \bullet q^t$	s_1^t	s_2^t	s_3^t	s_4^t	s_5^t	s_6^t
1	10.00	0.1000	0.1000	0.2000	0.1000	0.4500	0.0500
2	14.10	0.0681	0.1915	0.1752	0.0645	0.4667	0.0340
3	15.28	0.0654	0.0720	0.1767	0.0982	0.5563	0.0314
4	17.56	0.0547	0.0342	0.1731	0.1025	0.6059	0.0296
5	20.00	0.0450	0.0600	0.1600	0.0600	0.6500	0.0250

2. We will explain the trends that are built into the above tables. Think of the first 4 commodities as the consumption of various classes of *goods* in some economy while the last two commodities are the consumption of two classes of *services*. Think of the first good as *agricultural consumption*, which fluctuates around 1 and its price also fluctuates around 1. The quantity of the second good is *energy consumption* which trends up gently during the five periods with some minor fluctuations. However, note that the price of energy fluctuates wildly from period to period.[157] The third good is *traditional manufactures*. We have built in rather high inflation rates for this commodity for periods 2 and 3 which diminishes to a very low inflation rate by the end of our sample period.[158] The consumption of traditional manufactured goods is more or less static in our data set. The fourth commodity is *high technology manufactured goods*; e.g., computers, video cameras, compact disks, etc. We have the demand for these high tech commodities growing twelve times over our sample period while the final period price is only one tenth of the first period price. The fifth commodity is *traditional services*. The price trends for this commodity are similar to traditional manufactures, except that the period to period inflation rates are a bit higher. However, we have the demand for traditional services growing much more strongly than for traditional manufactures. Our final commodity is *high technology services*; e.g., telecommunications, wireless phones, internet services, stock market trading, etc. For this final commodity, we have the price trending downward very strongly to end up at 20% of the starting level while demand increases fivefold. The movements of prices and quantities in this artificial data set are more pronounced than the year to year movements that would be encountered in a typical country but they do illustrate the problem that is facing compilers of the Consumer Price

[157] This is an example of the price bouncing phenomenon noted by Szulc (1983). Note that the fluctuations in the price of energy that we have built into our data set are not that unrealistic: in the years 1998-2000, the price of a barrel of crude oil has fluctuated in the range $10 to $37 U.S.

[158] This corresponds roughly to the experience of most industrialized countries over the period starting in 1973 to the mid 1990's. Thus we are compressing roughly 5 years of price movement into one of our periods.

Index; namely, *year to year price and quantity movements are far from being proportional across commodities so the choice of index number formula will matter*.

3. Every price statistician is familiar with the *Laspeyres index* P_L defined by (3.5) in the main text of chapter 3 and the *Paasche index* P_P defined by (3.6) above. We list these indices in Table 3.3.4 along with the two unweighted indices that we considered: the *Carli index* defined by (3.64) and the *Jevons index* defined by (3.42). The indices in Table 3.3.4 compare the prices in period t with the prices in period 1; i.e., they are *fixed base indices*. Thus the period t entry for the Carli index, P_C, is simply the arithmetic mean of the 6 price relatives, $\sum_{i=1}^{6} (1/6)(p_i^t/p_i^1)$, while the period t entry for the Jevons index, P_J, is the geometric mean of the 6 price relatives, $\prod_{i=1}^{6} (p_i^t/p_i^1)^{1/6}$.

Table 3.3.4 The Fixed Base Laspeyres, Paasche, Carli and Jevons Indices

Period t	P_L	P_P	P_C	P_J
1	1.0000	1.0000	1.0000	1.0000
2	1.4200	1.3823	1.4000	1.2419
3	1.3450	1.2031	1.0500	0.9563
4	1.3550	1.0209	0.9167	0.7256
5	1.4400	0.7968	0.9833	0.6324

Note that by period 5, the spread between the fixed base Laspeyres and Paasche price indices is enormous: P_L is equal to 1.4400 while P_P is 0.7968, *a spread of about 81%*. Since both of these indices have exactly the same *theoretical* justification, it can be seen that the choice of index number formula matters a lot. The period 5 entry for the Carli index, 0.98333, falls between the corresponding Paasche and Laspeyres indices but the period 5 Jevons index, 0.63246, does not. Note that the Jevons index is always considerably below the corresponding Carli index. This will always be the case (unless prices are proportional in the two periods under consideration) because a geometric mean is always equal to or less than the corresponding arithmetic mean.[159]

4. It is of interest to recalculate the 4 indices listed in Table 3.3.4 above using *the chain principle* rather than the *fixed base principle* (see section E of chapter 3). Our expectation is that the spread between the Paasche and Laspeyres indices will be reduced by using the chain principle. These chain indices are listed in Table 3.3.5.

Table 3.3.5 Chain Laspeyres, Paasche, Carli and Jevons Indices

Period t	P_L	P_P	P_C	P_J
1	1.0000	1.0000	1.0000	1.0000
2	1.4200	1.3823	1.4000	1.2419
3	1.3646	1.2740	1.1664	0.9563
4	1.3351	1.2060	0.9236	0.7256
5	1.3306	1.1234	0.9446	0.6325

[159] This is the Theorem of the Arithmetic and Geometric Mean; see Hardy, Littlewood and Polyá (1934).

It can be seen comparing Tables 3.3.4 and 3.3.5 that chaining eliminated about 2/3 of the spread between the Paasche and Laspeyres indices. However, even the chained Paasche and Laspeyres indices differ by about 18% in period 5 so the choice of index number formula still matters. Note that chaining did not affect the Jevons index. This is an advantage of the index but the lack of weighting is a fatal flaw.[160] We would expect the "truth" to lie between the Paasche and Laspeyres indices and from Table 3.3.5, we see that the unweighted Jevons index is far below this acceptable range. Note that chaining did not affect the Carli index in a systematic way for our particular data set: in periods 3 and 4, the chained Carli is above the corresponding fixed base Carli but in period 5, the chained Carli is below the fixed base Carli.[161]

5. We turn now to a systematic comparison of all of *the asymmetrically weighted price indices* that were defined in Chapter 3 (with the exception of the Lloyd Moulton index which we will consider later). The *fixed base indices* are listed in Table 3.3.6. The fixed base *Laspeyres* and *Paasche indices*, P_L and P_P, are the same as those indices listed in Table 3.3.4 above. The *Palgrave index*, P_{PAL}, is defined by equation (3.73) in the main text of Chapter 3. The indices denoted by P_{GL} and P_{GP} are *the geometric Laspeyres and geometric Paasche indices*[162] which are special cases of the fixed weight geometric indices defined by Konüs and Byushgens; see (3.40) in Chapter 3. For *the geometric Laspeyres index*, P_{GL}, we let the weights α_i be the *base period expenditure shares*, s_i^1. This index should be considered an alternative to the fixed base Laspeyres index since each of these indices makes use of the same information set. For *the geometric Paasche index*, P_{GP}, we let the weights α_i be the *current period expenditure shares*, s_i^t. Finally, the index P_{HL} *is the harmonic Laspeyres index* that was defined by (3.77) in Chapter 3.

Table 3.3.6 Asymmetrically Weighted Fixed Base Indices

Period t	P_{PAL}	P_L	P_{GP}	P_{GL}	P_P	P_{HL}
1	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
2	1.6096	1.4200	1.4846	1.3300	1.3824	1.2542
3	1.4161	1.3450	1.3268	1.2523	1.2031	1.1346
4	1.5317	1.3550	1.3282	1.1331	1.0209	0.8732
5	1.6720	1.4400	1.4153	1.0999	0.7968	0.5556

[160] The problem with the evenly weighted geometric mean is that the price declines in high technology goods and services are given the same weighting as the price changes in the other 4 commodities (which have rising or stationary price changes) but the expenditure shares of the high technology commodities remain rather small throughout the 5 periods. Thus weighted price indices do not show the rate of overall price decrease that the unweighted Jevons index shows. These somewhat negative comments on the use of the unweighted geometric mean as an index number formula at higher levels of aggregation do not preclude its use at the very lowest level of aggregation where a strong axiomatic justification for the use of this formula can be given. If probability sampling is used at the lowest level of aggregation, then the unweighted geometric mean essentially becomes the logarithmic Laspeyres index.

[161] For many data sets, we would expect the chained Carli to be above the corresponding fixed base Carli; see Szulc (1983).

[162] Vartia (1978; 272) uses the terms *logarithmic Laspeyres* and *logarithmic Paasche* respectively.

By looking at the period 5 entries in Table 3.3.6, it can be seen that the spread between all of these fixed base asymmetrically weighted indices has increased to be even larger than our earlier spread of 81% between the fixed base Paasche and Laspeyres indices. In Table 3.3.6, the period 5 Palgrave index is about 3 times as big as the period 5 harmonic Laspeyres index, P_{HL}! Again, *this illustrates the point that due to the nonproportional growth of prices and quantities in most economies today, the choice of index number formula is very important*.

6. It is possible to explain why certain of the indices in Table 3.3.6 are bigger than others. It can be shown that a *weighted arithmetic mean* of n numbers is equal to or greater than the corresponding *weighted geometric mean* of the same n numbers which in turn is equal to or greater than the corresponding *weighted harmonic mean* of the same n numbers.[163] It can be seen that the three indices P_{PAL}, P_{GP} and P_P all use the current period expenditure shares s_i^t to weight the price relatives (p_i^t/p_i^1) but P_{PAL} is a weighted *arithmetic* mean of these price relatives, P_{GP} is a weighted *geometric* mean of these price relatives and P_P is a weighted *harmonic* mean of these price relatives. Thus by Schlömilch's inequality, we must have:[164]

$$(3.3.1)\quad P_{PAL} \geq P_{GP} \geq P_P \,.$$

Viewing Table 3.3.6, it can be seen that the inequalities (3.3.1) hold for each period. It can also be verified that the three indices P_L, P_{GL} and P_{HL} all use the base period expenditure shares s_i^1 to weight the price relatives (p_i^t/p_i^1) but P_L is a weighted *arithmetic* mean of these price relatives, P_{GL} is a weighted *geometric* mean of these price relatives and P_{HL} is a weighted *harmonic* mean of these price relatives. Thus by Schlömilch's inequality, we must have:[165]

$$(3.3.2)\quad P_L \geq P_{GL} \geq P_{HL} \,.$$

Viewing Table 3.3.6, it can be seen that the inequalities (3.3.2) hold for each period.

7. We continue with our systematic comparison of all of *the asymmetrically weighted price indices* that were defined in Chapter 3. These indices using the *chain principle* are listed in Table 3.3.7.

Table 3.3.7 Asymmetrically Weighted Indices Using the Chain Principle

Period t	P_{PAL}	P_L	P_{GP}	P_{GL}	P_P	P_{HL}
1	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
2	1.6096	1.4200	1.4846	1.3300	1.3824	1.2542
3	1.6927	1.3646	1.4849	1.1578	1.2740	0.9444
4	1.6993	1.3351	1.4531	1.0968	1.2060	0.8586
5	1.7893	1.3306	1.4556	1.0266	1.1234	0.7299

[163] This follows from Schlömilch's (1858) inequality; see Hardy, Littlewood and Polyá (1934; chapter 11).
[164] These inequalities were noted by Fisher (1922; 92) and Vartia (1978; 278).
[165] These inequalities were also noted by Fisher (1922; 92) and Vartia (1978; 278).

Viewing Table 3.3.7, it can be seen that although the use of the chain principle dramatically reduced the spread between the Paasche and Laspeyres indices P_P and P_L compared to the corresponding fixed base entries in Table 3.3.6, the spread between the highest and lowest asymmetrically weighted indices in period 5 (the Palgrave index P_{PAL} and P_{HL}) did not fall as much: the fixed base spread was 1.6720/0.5556 = 3.01 while the corresponding chain spread was 1.7893/0.7299 = 2.45. *Thus in this particular case, the use of the chain principle combined with the use of an index number formula that uses the weights of only one of the two periods being compared did not lead to a significant narrowing of the huge differences that these formulae generate using the fixed base principle. However, with respect to the Paasche and Laspeyres formulae, we find that chaining does significantly reduce the spread between these two indices.*

8. Is there an explanation for the results reported in the previous paragraph? It can be shown that all 6 of the indices that are found in the inequalities (3.3.1) and (3.3.2) approximate each other to the first order around an equal prices and quantities point. Thus with smooth trends in the data, we would expect all of the chain indices to more closely approximate each other than the fixed base indices because the changes in the individual prices and quantities would be smaller using the chain principle. This expectation is realized in the case of the Paasche and Laspeyres indices but not with the others. However, for some of the commodities in our data set, the trends in the prices and quantities are not smooth. In particular, the prices for our first two commodities (agricultural products and oil) bounce up and down. As noted by Szulc (1983), this will tend to cause the chain indices to have a wider dispersion than their fixed base counterparts. In order to determine if it is the bouncing prices problem that is causing some of the chained indices in Table 3.3.7 to diverge from their fixed base counterparts, we recomputed all of the indices in Tables 3.3.6 and 3.3.7 but excluding commodities 1 and 2 from the computations. The results of excluding these bouncing commodities may be found in Tables 3.3.8 and 3.3.9.

Table 3.3.8 Asymmetrically Weighted Fixed Base Indices for Commodities 3-6

Period t	P_{PAL}	P_L	P_{GP}	P_{GL}	P_P	P_{HL}
1	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
2	1.2877	1.2500	1.2621	1.2169	1.2282	1.1754
3	1.4824	1.4313	1.3879	1.3248	1.2434	1.1741
4	1.6143	1.5312	1.4204	1.3110	1.0811	0.9754
5	1.7508	1.5500	1.4742	1.1264	0.7783	0.5000

Table 3.3.9 Asymmetrically Weighted Chained Indices for Commodities 3-6

Period t	P_{PAL}	P_L	P_{GP}	P_{GL}	P_P	P_{HL}
1	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
2	1.2877	1.2500	1.2621	1.2169	1.2282	1.1754
3	1.4527	1.4188	1.4029	1.3634	1.3401	1.2953
4	1.5036	1.4640	1.4249	1.3799	1.3276	1.2782

5	1.4729	1.3817	1.3477	1.2337	1.1794	1.0440

It can be seen that excluding the bouncing price commodities does cause the chain indices to have a much narrower spread than their fixed base counterparts. *Thus our conclusion is that if the underlying price and quantity data is subject to reasonably smooth trends over time, then the use of chain indices will narrow considerably the dispersion in the asymmetrically weighted indices.* We now turn our attention to index number formulae that use weights from both periods in a symmetric or even handed manner.

9. Symmetrically weighted indices can be decomposed into two classes: *superlative indices* and *other symmetrically weighted indices*. Superlative indices have a close connection to economic theory; i.e., as we saw in sections H.3 to H.5 of Chapter 3, a superlative index is exact for a representation of the consumer's preference function or the dual unit cost function that can provide a second order approximation to arbitrary (homothetic) preferences. We considered 4 superlative indices in Chapter 4:

- the *Fisher ideal price index* P_F defined by (3.12);
- the *Walsh price index* P_W defined by (3.19) (this price index also corresponds to the quantity index Q^1 defined by (3.114) in Chapter 3);
- the *Törnqvist-Theil price index* P_T defined by (3.43) or (3.66) and
- the *implicit Walsh price index* P_{IW} that corresponds to the Walsh quantity index Q_W defined by (3.53) (this is also the index P^1 defined by (3.119) in Chapter 3).

These 4 symmetrically weighted superlative price indices are listed in Table 3.3.8 using the fixed base principle. We also list in this table two symmetrically weighted (but not superlative) price indices:[166]

- the Marshall Edgeworth price index P_{ME} defined by (3.18) and
- the Drobisch price index P_D defined above (3.12).

Table 3.3.10 Symmetrically Weighted Fixed Base Indices

Period t	$\mathbf{P_T}$	$\mathbf{P_{IW}}$	$\mathbf{P_W}$	$\mathbf{P_F}$	$\mathbf{P_D}$	$\mathbf{P_{ME}}$
1	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
2	1.4052	1.4015	1.4017	1.4011	1.4012	1.4010
3	1.2890	1.2854	1.2850	1.2721	1.2741	1.2656
4	1.2268	1.2174	1.2193	1.1762	1.1880	1.1438
5	1.2477	1.2206	1.1850	1.0712	1.1184	0.9801

Note that the Drobisch index P_D is always equal to or greater than the corresponding Fisher index P_F. This follows from the facts that the Fisher index is the geometric mean

[166] Diewert (1978; 897) showed that the Drobisch Sidgwick Bowley price index approximates any superlative index to the second order around an equal price and quantity point; i.e., P_{SB} is a *pseudo-superlative index*. Straightforward computations show that the Marshall Edgeworth index P_{ME} is also pseudo-superlative.

of the Paasche and Laspeyres indices while the Drobisch index is the arithmetic mean of the Paasche and Laspeyres indices and an arithmetic mean is always equal to or greater than the corresponding geometric mean. Comparing the fixed base asymmetrically weighted indices, Table 3.3.6, with the symmetrically weighted indices, Table 3.3.10, *it can be seen that the spread between the lowest and highest index in period 5 is much less for the symmetrically weighted indices*. The spread was 1.6720/0.5556 = 3.01 for the asymmetrically weighted indices but only 1.2477/0.9801 = 1.27 for the symmetrically weighted indices. If we restrict ourselves to the superlative indices listed for period 5 in Table 3.3.10, then this spread is further reduced to 1.2477/1.0712 = 1.16; i.e., the spread between the fixed base superlative indices is "only" 16% compared to the fixed base spread between the Paasche and Laspeyres indices of 81% (1.4400/0.7968 = 1.81). We expect to further reduce the spread between the superlative indices by using the chain principle.

10. We recompute the symmetrically weighted indices using the chain principle. The results may be found in Table 3.3.9.

Table 3.3.11 Symmetrically Weighted Indices Using the Chain Principle

Period t	P_T	P_{IW}	P_W	P_F	P_D	P_{ME}
1	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
2	1.4052	1.4015	1.4017	1.4011	1.4012	1.4010
3	1.3112	1.3203	1.3207	1.3185	1.3193	1.3165
4	1.2624	1.2723	1.2731	1.2689	1.2706	1.2651
5	1.2224	1.2333	1.2304	1.2226	1.2270	1.2155

A quick glance at Table 3.3.11 shows that *the combined effect of using both the chain principle as well as symmetrically weighted indices is to dramatically reduce the spread between all indices constructed using these two principles*. The spread between all of the symmetrically weighted indices in period 5 is only 1.2333/1.2155 = 1.015 or 1.5% and the spread between the 4 superlative indices in period 5 is an even smaller 1.2333/1.2224 = 1.009 or about 0.1%. The spread in period 5 between the two most commonly used superlative indices, the Fisher P_F and the Törnqvist P_T, is truly tiny: 1.2226/1.2224 = 0.0002.[167]

10. The results listed in Table 3.3.11 reinforce the numerical results tabled in Hill (2000) and Diewert (1978; 894): *the most commonly used chained superlative indices will generally give approximately the same numerical results*.[168] In particular, the chained Fisher, Törnqvist and Walsh indices will generally approximate each other very closely.

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[167] However, in other periods the differences were larger. On average over the last 4 periods, the chain Fisher and the chain Törnqvist indices differed by 0.0025 percentage points.

[168] More precisely, the superlative quadratic mean of order r price indices P^r defined by (3.116) and the implicit quadratic mean of order r price indices P^{r*} defined by (3.113) will generally closely approximate each other provided that r is in the interval $0 \le r \le 2$.

12. We now turn our attention to the differences between superlative indices and their counterparts that are constructed in two stages of aggregation; see section H.7 of Chapter 3 for a discussion of the issues and a listing of the formulae used. In our artificial data set, we will first aggregate the first 4 commodities into a *goods aggregate* and the last 2 commodities into a *services aggregate*. In the second stage of aggregation, the goods and services components will be aggregated into an all items index.

13. We report the results in Table 3.3.12 for our two stage aggregation procedure using period 1 as the *fixed base* for the Fisher index P_F, the Törnqvist index P_T and the Walsh and implicit Walsh indexes, P_W and P_{IW}.

Table 3.3.12 Fixed Base Superlative Single Stage and Two Stage Indices

Period t	P_F	P_{F2S}	P_T	P_{T2S}	P_W	P_{W2S}	P_{IW}	P_{IW2S}
1	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
2	1.4011	1.4004	1.4052	1.4052	1.4017	1.4015	1.4015	1.4022
3	1.2721	1.2789	1.2890	1.2872	1.2850	1.2868	1.2854	1.2862
4	1.1762	1.2019	1.2268	1.2243	1.2193	1.2253	1.2174	1.2209
5	1.0712	1.1286	1.2477	1.2441	1.1850	1.2075	1.2206	1.2240

Viewing Table 3.3.12, it can be seen that the fixed base single stage superlative indices generally approximate their fixed base two stage counterparts fairly closely with the exception of the Fisher formula. The divergence between the single stage Fisher index P_F and its two stage counterpart P_{F2S} in period 5 is 1.1286/1.0712 = 1.05 or 5%. The other divergences are 2% or less.

14. Using *chain indices*, we report the results in Table 3.3.13 for our two stage aggregation procedure. Again, the single stage and their two stage counterparts are listed for the Fisher index P_F, the Törnqvist index P_T and the Walsh and implicit Walsh indexes, P_W and P_{IW}.

Table 3.3.13 Chained Superlative Single Stage and Two Stage Indices

Period t	P_F	P_{F2S}	P_T	P_{T2S}	P_W	P_{W2S}	P_{IW}	P_{IW2S}
1	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
2	1.4011	1.4004	1.4052	1.4052	1.4017	1.4015	1.4015	1.4022
3	1.3185	1.3200	1.3112	1.3168	1.3207	1.3202	1.3203	1.3201
4	1.2689	1.2716	1.2624	1.2683	1.2731	1.2728	1.2723	1.2720
5	1.2226	1.2267	1.2224	1.2300	1.2304	1.2313	1.2333	1.2330

Viewing Table 3.3.13, it can be seen that *the chained single stage superlative indices generally approximate their fixed base two stage counterparts very closely indeed*. The divergence between the chained single stage Törnqvist index P_T and its two stage counterpart P_{T2S} in period 5 is 1.2300/1.2224 = 1.006 or 0.6%. The other divergences are all less than this. Given the large dispersion in period to period price movements, these two stage aggregation errors are not large.

15. The next formula that we illustrate using our artificial data set is the Lloyd Moulton index P_{LM} defined by (3.155) in Chapter 3. Recall that this formula requires an estimate for the parameter σ, *the elasticity of substitution* between all commodities being aggregated. Recall also that if σ equals 0, then the Lloyd Moulton index collapses down to the ordinary Laspeyres index, P_L. When σ equals 1, the Lloyd Moulton index is not defined but it can be shown that the limit of $P_{LM\sigma}$ as σ approaches 1 is P_{GL}, the geometric Laspeyres index or the logarithmic Laspeyres index with base period shares as weights. This index uses the same basic information as the fixed base Laspeyres index P_L and so it is a possible alternative index for CPI compilers to use. As was shown by Shapiro and Wilcox (1997)[169], the Lloyd Moulton index may be used to approximate a superlative index using the same information that is used in the construction of a fixed base Laspeyres index provided that we have an estimate for the parameter σ. We will test this methodology out using our artificial data set. The superlative index that we choose to approximate is the chain Fisher index[170] (which approximates the other chained superlative indices listed in Table 3.3.11 very closely). The chained Fisher index P_F is listed in column 2 of Table 3.3.14 along with the fixed base Lloyd Moulton indices $P_{LM\sigma}$ for σ equal to 0 (this reduces to the fixed base Laspeyres index P_L), 0.2, 0.3, 0.4, 0.5, 0.6, 0.7, 0.8 and 1 (which is the fixed base geometric index P_{GL}). Note that the Lloyd Moulton indices steadily *decrease* as we *increase* the elasticity of substitution σ.[171]

Table 3.3.14 Chained Fisher and Fixed Base Lloyd Moulton Indices

Period	P_F	P_{LM0}	$P_{LM.2}$	$P_{LM.3}$	$P_{LM.4}$	$P_{LM.5}$	$P_{LM.6}$	$P_{LM.7}$	$P_{LM.8}$	P_{LM1}
1	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
2	1.4011	1.4200	1.4005	1.3910	1.3818	1.3727	1.3638	1.3551	1.3466	1.3300
3	1.3185	1.3450	1.3287	1.3201	1.3113	1.3021	1.2927	1.2831	1.2731	1.2523
4	1.2689	1.3550	1.3172	1.2970	1.2759	1.2540	1.2312	1.2077	1.1835	1.1331
5	1.2226	1.4400	1.3940	1.3678	1.3389	1.3073	1.2726	1.2346	1.1932	1.0999

Viewing Table 3.3.14, it can be seen that *no single choice* of the elasticity of substitution σ will lead to a Lloyd Moulton price index $P_{LM\sigma}$ that will *closely* approximate the chained Fisher index P_F for periods 2,3,4 and 5. To approximate P_F in period 2, we should choose σ close to 0.1; to approximate P_F in period 3, we should choose σ close to 0.3; to approximate P_F in period 4, we should choose σ between 0.4 and 0.5 and to approximate P_F in period 5, we should choose σ between 0.7 and 0.8.[172]

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[169] Alterman, Diewert and Feenstra (1999) also used this methodology in the context of estimating superlative international trade price indices.

[170] Since there is still a considerable amount of dispersion among the fixed base superlative indices and practically no dispersion between the chained superlative indices, we take the Fisher chain index as our target rather than any of the fixed base superlative indices.

[171] This follows from Schlömilch's (1858) inequality again.

[172] Unfortunately, for this data set, neither the fixed base Laspeyres index $P_L = P_{LM0}$ nor the fixed base weighted geometric index $P_{GL} = P_{LM1}$ are very close to the chain Fisher index for all periods. For less extreme data sets, the fixed base Laspeyres and fixed base geometric indices will be closer to the chained Fisher index.

16. We repeat the computations for the Lloyd Moulton indices that are listed in Table 3.3.14 except that we now use the *chain principle* to construct the Lloyd Moulton indices; see Table 3.3.15. Again, we are trying to approximate the *chained Fisher price index* P_F which is listed as column 2 in Table 3.3.15. In Table 3.3.15, P_{LM0} is the chained Laspeyres index and P_{LM1} is the chained geometric Laspeyres or geometric index using the expenditure shares of the previous period as weights.

Table 3.3.15 Chained Fisher and Chained Lloyd Moulton Indices

Period	$\mathbf{P_F}$	$\mathbf{P_{LM0}}$	$\mathbf{P_{LM.2}}$	$\mathbf{P_{LM.3}}$	$\mathbf{P_{LM.4}}$	$\mathbf{P_{LM.5}}$	$\mathbf{P_{LM.6}}$	$\mathbf{P_{LM.7}}$	$\mathbf{P_{LM.8}}$	$\mathbf{P_{LM1}}$
1	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
2	1.4011	1.4200	1.4005	1.3910	1.3818	1.3727	1.3638	1.3551	1.3466	1.3300
3	1.3185	1.3646	1.3242	1.3039	1.2834	1.2628	1.2421	1.2212	1.2002	1.1578
4	1.2689	1.3351	1.2882	1.2646	1.2409	1.2171	1.1932	1.1692	1.1452	1.0968
5	1.2226	1.3306	1.2702	1.2400	1.2097	1.1793	1.1488	1.1183	1.0878	1.0266

Viewing Table 3.3.15, it can be seen that again *no single choice* of the elasticity of substitution σ will lead to a Lloyd Moulton price index $P_{LM\sigma}$ that will *closely* approximate the chained Fisher index P_F for all periods. To approximate P_F in period 2, we should choose σ close to 0.1; to approximate P_F in period 3, we should choose σ close to 0.2; to approximate P_F in period 4, we should choose σ between 0.2 and 0.3 and to approximate P_F in period 5, we should choose σ between 0.3 and 0.4. However, it should be noted that *if we choose σ equal to 0.3 and use the chained Lloyd Moulton index $P_{LM.3}$ to approximate the chained Fisher index P_F, we will have a much better approximation than that provided by either the chained Laspeyres index* (see P_{LM0} in column 3 of Table 3.3.15) *or the fixed base Laspeyres index* (see P_{LM0} in column 3 of Table 3.3.14).[173] Hence our tentative conclusions on the use of the Lloyd Moulton index to approximate superlative indices are:

- the elasticity of substitution parameter σ which appears in the Lloyd Moulton formula is *unlikely to remain constant over time* and hence it will be necessary for statistical agencies to update their estimates of σ at regular intervals and
- the use of the Lloyd Moulton index as a *real time preliminary estimator* for a chained superlative index seems warranted, provided that the statistical agency can provide estimates for chained superlative indices on a delayed basis. *The Lloyd Moulton index would provide a useful supplement to the traditional fixed base Laspeyres price index.*

17. The final formulae we illustrate using our artificial data set are the *additive percentage change decompositions* for the Fisher ideal index that were discussed in

[173] For this particular data set, the fixed base or chained geometric indices using either the expenditure weights of period 1 (see the last column of Table 3.3.14) or using the weights of the previous period (see the last column of Table 3.3.15) do not approximate the chained Fisher index very closely. However, for less extreme data sets, the use of chained Laspeyres or geometric indices may approximate a chained superlative index adequately.

section F.8 of Chapter 3. We will first decompose the *chain links* for the Fisher price index using the formulae (3.60) to (3.62) listed in section F.8. The results of the decomposition are listed in Table 3.3.16. Thus $P_F - 1$ is *the percentage change in the Fisher ideal chain link* going from period t − 1 to t and *the decomposition factor* $v_{Fi}\Delta p_i = v_{Fi}(p_i^t - p_i^{t-1})$ is the contribution to the total percentage change of the change in the ith price from p_i^{t-1} to p_i^t for i = 1,2,…,6.

Table 3.3.16 An Additive Percentage Change Decomposition of the Fisher Index

Period t	**$P_F - 1$**	**$v_{F1}\Delta p_1$**	**$v_{F2}\Delta p_2$**	**$v_{F3}\Delta p_3$**	**$v_{F4}\Delta p_4$**	**$v_{F5}\Delta p_5$**	**$v_{F6}\Delta p_6$**
2	0.4011	0.0176	0.1877	0.0580	-0.0351	0.1840	-0.0111
3	-0.0589	-0.0118	-0.1315	0.0246	-0.0274	0.0963	-0.0092
4	-0.0376	-0.0131	-0.0345	0.0111	-0.0523	0.0635	-0.0123
5	-0.0365	0.0112	0.0316	0.0000	-0.0915	0.0316	-0.0194

Viewing Table 3.3.16, it can be seen that the price index going from period 1 to 2 grew about 40% and the major contributors to this change were the increases in the price of commodity 2, energy (18.77%) and in commodity 5, traditional services (18.4%). The increase in the price of traditional manufactured goods, commodity 3, contributed 5.8% to the overall increase of 40.11%. The decreases in the prices of high technology goods (commodity 4) and high technology services (commodity 6) offset the other increases by −3.51% and −1.11% going from period 1 to 2. Going from period 2 to 3, the overall change in prices was negative: −5.89%. The reader can read across the third row of Table 3.3.16 to see what was the contribution of the 6 component price changes to the overall price change. It can be seen that a big price change in a particular component i combined with a big expenditure share in the two periods under consideration will lead to a big decomposition factor, v_{Fi}.

18. Our final set of computations we illustrate using our artificial data set is the *additive percentage change decomposition* for the Fisher ideal index that is due to Van Ijzeren (1987; 6) that was mentioned in section F.8 of Chapter 3. The *price* counterpart to the *additive decomposition* for a *quantity* index, equation (3.49) in Chapter 3, is:

$$(3.3.3)\quad P_F(p^0,p^1,q^0,q^1) = \sum_{i=1}^n q_{Fi}{}^*p_i^1 / \sum_{i=1}^n q_{Fi}{}^*p_i^0$$

where the reference quantities need to be defined somehow. Van Ijzeren (1987; 6) showed that the following reference weights provided an *exact additive representation for the Fisher ideal price index*:

$$(3.3.4)\quad q_{Fi}{}^* \equiv (1/2)q_i^0 + (1/2)q_i^1/Q_F(p^0,p^1,q^0,q^1)\ ; \qquad i = 1,2,\ldots,6$$

where Q_F is the overall Fisher quantity index. Thus using the Van Ijzeren quantity weights $q_{Fi}{}^*$, we obtain the *following Van Ijzeren additive percentage change decomposition for the Fisher price index*:

$$(3.3.5)\quad P_F(p^0,p^1,q^0,q^1) - 1 = \{\sum_{i=1}^6 q_{Fi}{}^*p_i^1 / \sum_{m=1}^6 q_{Fi}{}^*p_m^0\} - 1$$

$$= \{\sum_{i=1}^{6} q_{Fi}{*}p_i^{1} - \sum_{m=1}^{6} q_{Fi}{*}p_i^{0}\} / \sum_{m=1}^{6} q_{Fi}{*}p_m^{0}$$
$$= \sum_{i=1}^{6} v_{Fi}{*}\{p_i^{1} - p_i^{0}\}$$

where the *Van Ijzeren weight* for commodity i, v_{Fi}*, is defined as

(3..3.6) $v_{Fi}{*} \equiv q_{Fi}{*} / \sum_{m=1}^{n} q_{Fi}{*}p_m^{0}$; i = 1,…,6.

We will again decompose the *chain links* for the Fisher price index using the formulae (3.3.4) to (3.3.6) listed above. The results of the decomposition are listed in Table 3.3.17. Thus $P_F - 1$ is *the percentage change in the Fisher ideal chain link* going from period t − 1 to t and *the Van Ijzeren decomposition factor* $v_{Fi}{*}\Delta p_i$ is the contribution to the total percentage change of the change in the ith price from p_i^{t-1} to p_i^{t} for i = 1,2,…,6.

Table 3.3.17 Van Ijzeren's Decomposition of the Fisher Price Index

Period t	$P_F - 1$	$v_{F1}{*}\Delta p_1$	$v_{F2}{*}\Delta p_2$	$v_{F3}{*}\Delta p_3$	$v_{F4}{*}\Delta p_4$	$v_{F5}{*}\Delta p_5$	$v_{F6}{*}\Delta p_6$
2	0.4011	0.0178	0.1882	0.0579	-0.0341	0.1822	-0.0109
3	-0.0589	-0.0117	-0.1302	0.0243	-0.0274	0.0952	-0.0091
4	-0.0376	-0.0130	-0.0342	0.0110	-0.0521	0.0629	-0.0123
5	-0.0365	0.0110	0.0310	0.0000	-0.0904	0.0311	-0.0191

Comparing the entries in Tables 3.3.16 and 3.3.17, it can be seen that the differences between the Diewert and Van Ijzeren decompositions of the Fisher price index are *very small*. The maximum absolute difference between the $v_{Fi}\Delta p_i$ and $v_{Fi}{*}\Delta p_i$ is only 0.0018 (about 0.2 percentage points) and the average absolute difference is 0.0003. This is somewhat surprising given the very different nature of the two decompositions.[174] As was mentioned in section F.8 of Chapter 3, the Van Ijzeren decomposition of the chain Fisher *quantity* index is used by the Bureau of Economic Analysis in the U.S.

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[174] However, Reinsdorf, Diewert and Ehemann (2001) show that the terms in the two decompositions approximate each other to the second order around any point where the two price vectors are equal and where the two quantity vectors are equal.

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