Equilibria in Second Price Auctions with Participation Costs

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Abstract

We investigate equilibria of sealed-bid second price auctions with bidder participation costs in the independent private values environment. We focus on equilibria in which bidders use cut-off strategies (bid the valuation if it is greater than a certain cut-off point, otherwise do not participate), since if a bidder finds participating optimal, she cannot do better than bidding her valuation. When the bidders are symmetric, the concavity (respectively, strict convexity) of the c.d.f. from which the valuations are drawn is a sufficient condition for uniqueness (respectively, multiplicity) within this class. We also study a special case with asymmetric bidders and show that concavity/convexity plays a similar role.

JEL Classification: C62, C72, D44, D82

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1 Introduction

In this paper, we investigate equilibria of sealed-bid second price auctions (also called Vickrey auctions) with bidder participation costs in the indepen-

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dent private values environment.[1] When the bidders are ex-ante symmetric, i.e., their valuations are drawn from the same distribution, the literature, in general, has focused on the unique symmetric equilibrium in which each bidder bids her valuation if it is greater than the cut-off point (common to all bidders), otherwise chooses not to participate.[2] We want to know when, if at all, this is the only equilibrium. We also want to identify sufficient conditions for existence of asymmetric equilibria.

It is not a weakly dominant strategy for a bidder, as it is the case when there is no participation cost, to always bid her valuation. However, given the strategies of others, if a bidder finds participating optimal, she cannot do better than bidding her valuation. Therefore, we focus on equilibria in which each bidder uses a *cut-off strategy* (bid her valuation if it is greater than a certain cut-off point, otherwise do not participate), so, for example, results about uniqueness of equilibria refer to uniqueness within this class. When the bidders are symmetric, there is a unique symmetric equilibrium. We show that if the valuations of the bidders are distributed according to a *concave* cumulative distribution function (c.d.f.), then there is no other equilibrium.[3] If the c.d.f. is *strictly convex*, on the other hand, then there will *always* be asymmetric equilibria.[4] In particular, there will always be "two cut-off" equilibria: Arbitrarily divide the set of bidders into two groups. There is an equilibrium in which bidders in the same group use the same cut-off that is different than the one used by the other group. Furthermore, if the c.d.f. is log-concave, then at most two cut-off points are used in any equilibrium.

The existence of asymmetric equilibria has important consequences for both efficiency of the auction mechanism and the seller's revenue. To begin with, asymmetric equilibria will necessarily be ex-post inefficient: the bidder with the highest valuation does not always get the object. Secondly, Stegeman (1996) considers ex-ante efficient mechanisms (maximizing expected total surplus net of participation costs) in the same environment and shows that the second price auction will have an ex-ante efficient equilibrium, whereas

[1] In the analysis below, bidder "participation cost" can be replaced by "entry fee" charged by the seller without any change in the formal model or results. Accordingly, we will use both interpretations in our discussions.

[2] The only exception that we are aware of is Stegeman (1996), where there is also an example with two equilibria.

[3] Concavity of the c.d.f. may be even more restrictive than it seems. See Remark 3 in Section 3.

[4] See Remark 2 in Section 3 on the effect of a binding reserve price.

the first price auction has an efficient equilibrium if and only if the second price auction has an efficient *symmetric* equilibrium. So, by finding sufficient conditions for uniqueness of equilibrium (necessarily symmetric), we also identify environments in which both first and second price auctions are ex-ante efficient mechanisms. Thirdly, consider the entry fee interpretation. The result that, under certain assumptions, the second price auction with appropriately chosen reserve price and entry fee is the seller's optimal mechanism depends on bidders playing the symmetric equilibrium. If the seller uses an entry fee, then her revenue from an asymmetric equilibrium will (generically) be lower than the maximum possible, i.e., revenue from the symmetric equilibrium. Finally, in a repeated auction environment, the asymmetric equilibria of the stage game that we identify can be used to construct collusive repeated game equilibria.

We also look at a special case with asymmetric bidders. There are two groups of bidders, where bidders are symmetric within groups, and one group is "stronger" than the other, i.e., bidders in this group are more likely to have higher valuations. Besides the literal interpretation, this case may also endogenously arise during the cartel formation process among symmetric bidders, as in Tan and Yilankaya (2003). We concentrate on equilibria that are symmetric within groups, given that within group asymmetries pose similar issues that we analyzed in the symmetric bidders case. We show that the "intuitive" equilibrium, where the strong bidders are more likely to participate in the auction, always exists, and this type of equilibrium is unique when both c.d.f.'s are concave. When the weak bidders' c.d.f. is concave, there is never an equilibrium in which they are more likely to participate in the auction. When it is strictly convex, on the other hand, there will be such counterintuitive equilibria, as long as the participation cost is high enough.

In the next section we briefly describe the setup. We look at the cases of symmetric and asymmetric bidders in Sections 3 and 4, respectively. All the proofs are in the Appendix.

2 The Setup

There are $n \geq 2$ risk-neutral (potential) bidders. The valuation of bidder i is v_i, which is private and distributed on $[0, 1]$, independent of other bidders' valuations, with c.d.f. $F_i(.)$ that has continuous density $f_i(.)$.

The auction format is sealed-bid second price. There is a participation cost, common to all bidders, denoted by $c \in (0,1)$; bidders must incur c in order to be able to submit a bid. Bidders know their valuations before they decide whether to participate in the auction. Bidders do not know others' participation decisions when they make theirs.[5]

Let the feasible action set for any type of bidder be: $\{No\} \cup [0, \infty)$, where "No" denotes not participating; bidder i incurs the participation cost iff her action is different from "No". Let $b_i(v_i)$ denote i's strategy.

If a bidder finds participating optimal, she cannot do better than bidding her valuation.[6] Therefore, we focus on the equilibria in which each bidder uses a *cut-off strategy*, where she bids her valuation if it is greater than a certain cut-off point, otherwise she does not participate.[7] That is for each bidder,

$$b_i(v_i) = \begin{cases} No & \text{if} \quad v_i \leq a_i \\ v_i & \text{if} \quad v_i > a_i \end{cases},$$

where, without loss of generality, $a_i \in [0,1]$, and $a_i = 1$ means that bidder i does not participate in the auction whatever her valuation is. Since we restrict attention to cut-off strategies, from now on we focus only on cut-off points. Notice that in every equilibrium $a_i \geq c$ for all i. Moreover, whenever $a_i < 1$, it is determined by an indifference condition (between participating and not participating) for type-a_i bidder. If $a_i = 1$, then bidder i's payoff when $v_i = 1$ must be nonpositive.

Unless specified, results below are valid for all $c \in (0,1)$.

3 Symmetric Bidders

In this section we analyze the case in which bidders' valuations are drawn from the same distribution function, i.e., $F_i(.) = F(.)\ \forall i$. It is well known that there is a unique symmetric equilibrium; we include this result here for completeness.

[5]See, for example, Samuelson (1985) and Matthews (1995) for auctions with participation costs or entry fees.

[6]See, for example, Matthews (1995).

[7]Blume and Heidhues (2003) characterize all equilibria of the second price auction both with and without a reserve price. See Remark 1 below about the effect of the participation cost on the possible existence of other equilibria.

Proposition 1 *There exists a unique symmetric equilibrium in which $a_i = a^* \ \forall i$, where*

$$a^* F(a^*)^{n-1} = c.$$

When $F(.)$ is concave this is the only equilibrium: No asymmetric equilibrium exists.

Proposition 2 *If $F(.)$ is concave, then no asymmetric equilibrium exists.*

On the other hand, when $F(.)$ is strictly convex there will always be asymmetric equilibria. In particular, if we partition the set of bidders into two groups in *any* way, then there will be an equilibrium where all the bidders within a group use the same cut-off that is different than the cut-off used by bidders in the other group. Furthermore, when $F(.)$ is log-concave there is no equilibria in which three or more cut-off points are used. Thus, when $F(.)$ is both strictly convex and log-concave, the set of equilibria is characterized by Propositions 1 and 3*i*.

Proposition 3 *i) If $F(.)$ is strictly convex, then, for any $k \in \{1, 2, ..., n-1\}$, there exists an asymmetric equilibrium in which k bidders use cut-off point a, $n-k$ bidders use cut-off point b, where $a < a^* < b$,*

$$aF(a)^{k-1}F(b)^{n-k} - c = 0, \tag{1}$$

and

$$F(b)^{n-k-1}[aF(a)^k + \int_a^b F(y)^k dy] - c \leq 0, \tag{2}$$

and (2) holds with equality whenever $b < 1$.

ii) If $F(.)$ is log-concave, then there is no equilibrium in which three or more (distinct) cut-off points are used.

To illustrate the role played by concavity/convexity of the c.d.f. on the possible existence of asymmetric equilibria, consider the case of two-bidders. Denote the first (respectively, second) bidder's cut-off by a (respectively, b), where, without loss of generality, $b \geq a$. Expected net-payoff of the first bidder must be zero when her type is a:

$$aF(b) - c = 0. \tag{3}$$

Similarly, for the second bidder the following must be true (with equality whenever $b < 1$):

$$bF(a) + \int_a^b (b-y)dF(y) - c \leq 0,$$

where the first term on the left-hand side is the payoff to type-b bidder 2 when bidder 1 does not participate and the second term is the payoff when bidder 1's valuation is between a and b. After integration by parts, the inequality becomes

$$aF(a) + \int_a^b F(y)dy - c \leq 0. \tag{4}$$

Obviously, (3) and (4) admit a symmetric equilibrium cut-off point, $a = b = \bar{a}$, determined by $\bar{a}F(\bar{a}) - c = 0$. To see whether there exists an asymmetric equilibrium, using (3) and (4), let $\alpha(\beta) = \frac{c}{F(\beta)}$ and

$$\pi(\beta) = \alpha(\beta)F(\alpha(\beta)) + \int_{\alpha(\beta)}^{\beta} F(y)dy - c,$$

where $\beta \in [\bar{a}, 1]$. Notice that $\pi(.)$ is the net-payoff to type-β bidder 2, taking into account the best response of bidder 1 to bidder 2's use of cut-off β, and also that it is continuous and $\pi(\bar{a}) = 0$.

An interior asymmetric equilibrium exists and given by $a = \alpha(\beta^*)$ and $b = \beta^*$ iff $\pi(\beta^*) = 0$ for $\beta^* \in (\bar{a}, 1)$. A corner asymmetric equilibrium exists, with $a = c$ and $b = 1$, iff $\pi(1) \leq 0$. Therefore, existence of asymmetric equilibria depends on the sign of $\pi(1)$ and whether $\pi(\beta^*) = 0$ for some $\beta^* \in (\bar{a}, 1)$.

Consider a concave $F(.)$. In this case $\pi(.)$ is strictly increasing, and hence $\pi(\beta) > 0$ for all $\beta \in (\bar{a}, 1]$. Therefore, an asymmetric (interior or corner) equilibrium cannot exist. Now, suppose that $F(.)$ is strictly convex. If $\pi(1) \leq 0$ (this happens for c large enough), then there is an equilibrium with $a = c$ and $b = 1$. If $\pi(1) > 0$, then there exists $\beta^* \in (\bar{a}, 1)$ such that $\pi(\beta^*) = 0$, i.e., an interior asymmetric equilibrium exists, since $\pi(\bar{a}) = 0$ and $\pi(.)$ is strictly decreasing at $\bar{a}$ when $F(.)$ is strictly convex.

Remarks:

1) Let r denote the reserve price. There is an equilibrium (in weakly dominated strategies) of the second price auction with $r = c = 0$ in which one of the bidders bids, say, 1, and others bid 0. Blume and Heidhues (2003) show that, when $r > 0$ (and $c = 0$), there is essentially a unique equilibrium in which each bidder bids her valuation whenever it is greater than the reserve price, as long as there are at least three bidders. When $c > 0$, equilibria in weakly dominated strategies may exist, and, interestingly enough, the concavity/convexity of the c.d.f. plays a similar role. Suppose $r = 0$ and $c > 0$, and consider the following strategy profile: One bidder bids 1 if her valuation is greater than c, otherwise she does not participate; other bidders never participate. For this profile to be an equilibrium, we only need to check that the highest possible payoff of nonparticipating bidders is nonpositive, i.e., $F(c) - c \leq 0$. It follows that, for any c, when $F(.)$ is convex this strategy profile is an equilibrium, and when $F(.)$ is strictly concave, it is not. Consider the following profile when $0 < r, c, r+c < 1$: One bidder bids 1 if her valuation is greater than $r+c$, otherwise she does not participate; other bidders never participate. Again, this is an equilibrium iff the highest possible payoff of nonparticipating bidders is nonpositive, i.e., $F(r+c)(1-r) - c \leq 0$. When $F(.)$ is concave, this is never the case, when $F(.)$ is strictly convex it depends on the magnitude of r and c.

2) Existence of a reserve price does not affect Propositions 1 and 2, except that the equation for the symmetric cut-off a^* becomes:

$$(a^* - r)F(a^*)^{n-1} = c.$$

With a reserve price, strict convexity of $F(.)$ is no longer sufficient for existence of multiple equilibria specified in Proposition 3 . It is not difficult to show that (following the current proof after adjusting the equations (1) and (2)), for any given r and c, a sufficient condition for existence of asymmetric equilibria is:

$$F(a^*) - (a^* - r)f(a^*) < 0,$$

where a^* is the symmetric cut-off defined above. We used strict convexity as a sufficient condition in Proposition 3, rather than the inequality above (with $r = 0$), for the result to hold for every c.

3) The assumption that the bidders valuations are bounded from above by 1 is without loss of generality. However, it is crucial that the lowest possible valuation of a bidder is zero for the uniqueness result in Proposition 2 to be meaningful. When the lowest possible bidder valuation is greater than zero, this will introduce convexity, even if $F(.)$ is concave when its domain is restricted to its support, and there will be asymmetric equilibria. Stegeman's (1996) example belongs to this case.

4) When $F(.)$ is neither concave nor convex, the existence of asymmetric equilibria depends on the magnitude of the participation cost. For example, suppose there are two bidders whose valuations are distributed according to $F(v) = v^3 - v^2 + v$. (Notice that $F''(.) > 0$ iff $v > \frac{1}{3}$.) It is not difficult to show that there exists an asymmetric equilibrium iff $c > .1875$.

5) When $F(.)$ is convex, but not log-concave, there may exist equilibria with three or more cut-off points. Let $n = 3, F(v) = \frac{v+v^5}{2}$ ($\frac{f(.)}{F(.)}$ is increasing iff $v \gtrsim 0.57$),and $c = 0.15$. There is an equilibrium in which bidders use (approximately) 0.3938, 0.8034, and 0.8622 as cut-offs.

4 Asymmetric Bidders

In this section we look at the case in which there are two groups of bidders. In particular, there are s "strong" bidders whose valuations are distributed with $G(.)$, and $n - s$ "weak" bidders whose valuations are distributed according to $F(.)$, where $s \in \{1, 2, ...n - 1\}$ and $G(v) < F(v)$ for all $v \in (0, 1)$.

We concentrate on equilibria that are symmetric within groups, i.e., every strong (respectively, weak) bidder uses the same cut-off point.[8] Denote the strong bidders' cut-off by a, and the weak bidders' by b.

We first show that the "intuitive" equilibrium, where the strong bidders are more likely to participate in the auction, always exists independent of the distribution functions. Moreover, if both c.d.f.'s are concave, then there is a unique intuitive equilibrium.

Proposition 4 *There exists an equilibrium with $b > a$. If $F(.)$ and $G(.)$ are concave, then there is a unique equilibrium with $b > a$.*

[8]The issue of within-group asymmetry is similar to that of the previous section, and thus ignored.

When $F(.)$ is concave, there is never an equilibrium in which weak bidders are more likely to participate in the auction. However, when $F(.)$ is strictly convex, as long as the participation cost is high enough, there will be such counterintuitive equilibria.

Proposition 5 *i) If $F(.)$ is concave, then there is no equilibrium with $a > b$.*

ii) If $F(.)$ is strictly convex, then $\exists c^* < 1$ s.t. there is an equilibrium with $a > b$ whenever $c > c^*$.

5 Appendix

Proof of Proposition 1. All the bidders will use the cut-off point $a^* \in (c, 1)$ given by

$$a^* F(a^*)^{n-1} = c.$$

Such a unique a^* exists, since $vF(v)^{n-1}$ is continuous, strictly increasing, and

$$cF(c)^{n-1} < c < 1F(1)^{n-1} = 1.$$

■

Proof of Proposition 2. Suppose there exists an asymmetric equilibrium. Fix one. Let $a_1 < a_2 < ... < a_k$ be the cut-off points used by bidders, and denote the number of bidders using a_i by n_i. We have

$$a_1 F(a_1)^{n_1 - 1} F(a_2)^{n_2} p = c$$

and

$$pF(a_2)^{n_2 - 1}[a_2 F(a_1)^{n_1} + \int_{a_1}^{a_2} (a_2 - y) dF(y)^{n_1}] \leq c,$$

or after integration by parts:

$$pF(a_2)^{n_2 - 1}[a_1 F(a_1)^{n_1} + \int_{a_1}^{a_2} F(y)^{n_1} dy] \leq c,$$

where p is the probability of valuations of bidders who use $a_3, ..., a_k$ be less than their corresponding cut-off points. Combining these:

$$a_1 F(a_1)^{n_1 - 1} F(a_2) \geq a_1 F(a_1)^{n_1} + \int_{a_1}^{a_2} F(y)^{n_1} dy >$$
$$a_1 F(a_1)^{n_1} + (a_2 - a_1) F(a_1)^{n_1} = a_2 F(a_1)^{n_1},$$

or

$$\frac{F(a_2)}{a_2} > \frac{F(a_1)}{a_1},$$

which is a contradiction with $F(.)$ being concave and $a_2 > a_1$. ■

Proof of Proposition 3. i) The indifference condition for k bidders who use a:

$$aF(a)^{k-1}F(b)^{n-k} - c = 0. \tag{5}$$

Notice that the curve representing this equation in (a, b) plane is decreasing, and it passes through $(a, b) = (a_*, 1)$ and (a^*, a^*), where $a \in [a_*, a^*]$, and $a^* > a_*$ are given by: $a_*F(a_*)^{k-1} - c = 0$, and $a^*F(a^*)^{n-1} - c = 0$.[9]

The indifference condition for $n - k$ bidders who use b:

$$F(b)^{n-k-1}[aF(a)^k + \int_a^b F(y)^k dy] - c = 0. \tag{6}$$

This also gives a decreasing curve that passes through (a^*, a^*). If $(a_*, 1)$ lies below this curve, then we have an equilibrium with cut-off points a_* and 1, and we are done. Assume otherwise, then. When $a = a_*$, the second curve is below the first one, and when $a = a^*$, two curves intersect. We need to show that these curves intersect at some $a \in (a_*, a^*)$ to prove the existence of the desired asymmetric equilibrium. For this, we only need to show that the second curve is steeper than the first one at $a = a^*$. From (5):

$$\left|\frac{db}{da}\right|_{(5)} = \frac{(k-1)af(a)F(b) + F(a)F(b)}{(n-k)aF(a)f(b)},$$

and from (6):

$$\left|\frac{db}{da}\right|_{(6)} = \frac{kaf(a)F(a)^{k-1}F(b)}{F(b)^{k+1} + (n-k-1)f(b)[aF(a)^k + \int_a^b F(y)^k dy]}.$$

At $a = b = a^*$, these are equal to, correspondingly,

$$\left|\frac{db}{da}\right|_{(5)} = \frac{(k-1)a^*f(a^*) + F(a^*)}{(n-k)a^*f(a^*)},$$

[9] We are abusing the notation here, since (a, b) denoted the solution to two indifference conditions.

and

$$\left|\frac{db}{da}\right|_{(6)} = \frac{ka^* f(a^*)}{F(a^*) + (n-k-1)a^* f(a^*)}.$$

Since $F(.)$ is strictly convex, $a^* f(a^*) > F(a^*)$, and the conclusion follows.

ii) If $n = 2$, then the claim is vacuously true. Let $n \geq 3$. Suppose there exists an equilibrium in which three or more cut-off points are used. Fix one. Let $a_1 < a_2 < ... < a_k$, $k \geq 3$, be the cut-off points used by bidders, and denote the number of bidders using a_i by n_i. We have

$$a_1 F(a_1)^{n_1 - 1} F(a_2)^{n_2} F(a_3)^{n_3} p = c, \tag{7}$$

$$F(a_2)^{n_2 - 1} F(a_3)^{n_3} p[a_1 F(a_1)^{n_1} + \int_{a_1}^{a_2} F(y)^{n_1} dy] = c, \tag{8}$$

and

$$a_3 F(a_1)^{n_1} F(a_2)^{n_2} F(a_3)^{n_3 - 1} p + F(a_2)^{n_2} F(a_3)^{n_3 - 1} p \int_{a_1}^{a_2} (a_3 - y) dF(y)^{n_1}$$
$$+ F(a_3)^{n_3 - 1} p \int_{a_2}^{a_3} (a_3 - y) dF(y)^{n_1 + n_2} \leq c,$$

where p is the probability that all the bidders who have cutoff points a_i, $i > 3$, do not participate. After integration by parts, the last condition becomes

$$F(a_2)^{n_2} F(a_3)^{n_3 - 1} p[a_1 F(a_1)^{n_1} + \int_{a_1}^{a_2} F(y)^{n_1} dy] + F(a_3)^{n_3 - 1} p \int_{a_2}^{a_3} F(y)^{n_1 + n_2} dy \leq c. \tag{9}$$

Combining (7) and (8) yields

$$\frac{F(a_2)^{n_2} F(a_3)^{n_3} p \int_{a_1}^{a_2} F(y)^{n_1} dy}{F(a_2) - F(a_1)} = c.$$

Similarly, combining (8) and (9) yields

$$\frac{F(a_3)^{n_3} p \int_{a_2}^{a_3} F(y)^{n_1 + n_2} dy}{F(a_3) - F(a_2)} \leq c.$$

Since $F(y) > F(a_2)$ for $y > a_2$, it follows that

$$\frac{\int_{a_2}^{a_3} F(y)^{n_1} dy}{F(a_3) - F(a_2)} < \frac{\int_{a_1}^{a_2} F(y)^{n_1} dy}{F(a_2) - F(a_1)}$$

or, equivalently,

$$\frac{\phi(x_3) - \phi(x_2)}{x_3 - x_2} < \frac{\phi(x_2) - \phi(x_1)}{x_2 - x_1}, \tag{10}$$

where $x_i = F(a_i)$ and

$$\phi(x) = \int_0^{F^{-1}(x)} F(y)^{n_1} dy.$$

Note that

$$\phi^{'}(x) = \frac{F(F^{-1}(x))^{n_1}}{f(F^{-1}(x))},$$

so that $\phi(.)$ is strictly increasing and convex, since $F(.)$ is strictly increasing and $\frac{F(.)}{f(.)}$ increasing ($F(.)$ is log-concave). This contradicts (10). ■

Proof of Proposition 4. In any such equilibrium, we have

$$aG(a)^{s-1}F(b)^{n-s} - c = 0, \tag{11}$$

$$F(b)^{n-s-1}[aG(a)^s + \int_a^b G(y)^s dy] - c \leq 0. \tag{12}$$

Define $a(b)$ from (11), and let $\widetilde{b}G(\widetilde{b})^{s-1}F(\widetilde{b})^{n-s} - c = 0$, i.e., $a(\widetilde{b}) = \widetilde{b}$, and

$$h(b) = F(b)^{n-s-1}[a(b)G(a(b))^s + \int_{a(b)}^b G(y)^s dy] - c.$$

Notice that $h(b)$ is continuous, $b \in [\widetilde{b}, 1]$,

$$\begin{aligned} h(\widetilde{b}) &= \widetilde{b}G(\widetilde{b})^s F(\widetilde{b})^{n-s-1} - c \\ &= \frac{G(\widetilde{b})}{F(\widetilde{b})} c - c < 0, \end{aligned}$$

and

$$h(1) = cG(c)^s + \int_c^1 G^s(y)dy - c.$$

Now, if $h(1) > 0$, then there is an equilibrium with $a < b < 1$. If $h(1) < 0$, then there is an equilibrium with $b = 1$ and $a = a(1) < 1$.

This equilibrium is unique if $F(.)$ and $G(.)$ are concave. Notice that (11) is decreasing. Substituting for c from (11), (12) becomes:

$$aG(a)^s + \int_a^b G(y)^s dy - aG(a)^{s-1}F(b) = 0.$$

We will show that this curve is increasing, and so it has a unique intersection with (11).

$$\frac{db}{da} = \frac{-asG(a)^{s-1}g(a) + F(b)[a(s-1)G(a)^{s-2}g(a) + G(a)^{s-1}]}{G(b)^s - aG(a)^{s-1}f(b)}.$$

The numerator is positive, since $F(b) > G(b) > G(a) \geq ag(a)$, where the last inequality follows from the concavity of $G(.)$. Moreover,

$$aG(a)^s + (b-a)G(b)^s - aG(a)^{s-1}F(b) > aG(a)^s + \int_a^b G(y)^s dy - aG(a)^{s-1}F(b) = 0,$$

$$bG(b)^s - aG(a)^{s-1}F(b) > aG(b)^s - aG(a)^s > 0,$$

or

$$G(b)^s > aG(a)^{s-1}\frac{F(b)}{b} \geq aG(a)^{s-1}f(b),$$

where the last inequality follows from the concavity of $F(.)$. Hence $\frac{db}{da} > 0$. ■

Proof of Proposition 5. i) Suppose there is such an equilibrium. Then

$$bF(b)^{n-s-1}G(a)^s - c = 0, \tag{13}$$

$$G(a)^{s-1}[bF(b)^{n-s} + \int_b^a F(y)^{n-s}dy] - c \leq 0.$$

Combining,

$$bF(b)^{n-s-1}G(a) - bF(b)^{n-s} \geq \int_b^a F(y)^{n-s}dy > (a-b)F(b)^{n-s},$$

or

$$\frac{G(a)}{a} > \frac{F(b)}{b},$$

which is a contradiction, since concavity of $F(.)$ implies that $\frac{F(b)}{b} \geq \frac{F(a)}{a} \geq \frac{G(a)}{a}$.

ii) Define $a(b)$ from (13), and notice that $b \in [b_1, b_2]$, where $b_1 F(b_1)^{n-s-1} - c = 0$ and $b_2 F(b_2)^{n-s-1} G(b_2)^s - c = 0$, so that $a(b_1) = 1$ and $a(b_2) = b_2$. Let

$$h(b) = G(a(b))^{s-1}[bF(b)^{n-s} + \int_b^{a(b)} F(y)^{n-s}dy] - c.$$

We have the required equilibrium iff $\exists b^* \in [b_1, b_2]$ with $h(b^*) = 0$. We know that

$$h(b_2) = b_2 F(b_2)^{n-s} G(b_2)^{s-1} - c > 0,$$

since $F(b_2) > G(b_2)$. We only need, then,

$$h(b_1) = b_1 F(b_1)^{n-s} + \int_{b_1}^1 F(y)^{n-s}dy - c < 0.$$

From its definition, b_1 is an (increasing) function of c. What we need is,

$$\tilde{h}(c) = b_1(c) F(b_1(c))^{n-s} + \int_{b_1(c)}^1 F(y)^{n-s}dy - c < 0.$$

Now, $\tilde{h}(1) = 0$ and $\tilde{h}'(1) > 0$ if $F(.)$ is strictly convex. Hence, $\exists c^* < 1$ s.t. $\tilde{h}(c) < 0$ whenever $c > c^*$. ■

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