Index-Number Tests and the Common-Scaling Social Cost-of-Living Index[1]

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Abstract

For a change in prices, the common-scaling social cost-of-living index is the equal scaling of every individual's expenditure level needed to restore the level of social welfare to its pre-change value. This index does not, in general, satisfy two standard index-number tests. The reversal test requires the index value for the reverse change to be the reciprocal of the original index. And the circular test requires the product of index values for successive price changes to be equal to the index value for the whole change. We show that both tests are satisfied if and only if the Bergson-Samuelson indirect social-welfare function is homothetic in prices, a condition which does not require individual preferences to be homothetic. If individual preferences *are* homothetic, however, stronger conditions on the Bergson-Samuelson indirect must be satisfied. Given these results, we ask whether the restrictions are empirically reasonable and find, in the case that individual preferences are not homothetic, that the restrictions make little difference to estimates of the index.

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Recently, Thomas Crossley and Krishna Pendakur [2009] have proposed a social cost-of-living index that is a departure from the standard ones. For a price change, their common-scaling social cost-of-living index is the equal scaling of every individual's expenditure level needed to restore the pre-change level of social welfare. Using a social-welfare function, the index takes account of a social attitude towards inequality of well-being when assessing the social cost of price changes.

Like the common-scaling index, the other general ethical index, due to Pollak [1980, 1981], uses a social-welfare function. The value of Pollak's index depends on a reference level of social welfare and is equal to the ratio of total expenditure needed to achieve that level after the change to total expenditure needed before the change. Blackorby and Donaldson [1983] investigate conditions under which the index is independent of the reference level of social welfare, given that the social-welfare function is additively separable.

The commonly used plutocratic index is the share-weighted value of individual cost-of-living indexes. It is equal to the ratio of the total cost of keeping each person at his or her pre-change utility level to total expenditures before the change. Given appropriate social-welfare and individual utility functions, the common-scaling index is the plutocratic index (see Section 2). The arithmetic mean of individual indexes is also used, but has no desirable ethical properties.

The Pollak and common-scaling indexes are generalizations of individual cost-of-living indexes, which can be defined as the ratio of expenditures needed to achieve a reference level of individual utility, parallel to Pollak's index, or as the scaling of individual expenditure need to preserve the pre-change utility level, parallel to the common-scaling index. Because cost-of-living adjustments are typically made with across-the-board percentage changes, the common-scaling index may be more attractive.

Suppose that an economy consists of n single adults, where n is a positive integer, with m goods, $m \geq 2$. Individual i has a continuous indirect utility function $V^i\colon \mathcal{R}^{m+1}_{++} \to \mathcal{R}$ which is increasing in expenditure, weakly decreasing in prices, and homogeneous of degree zero.

The social-welfare function $W\colon V^1(\mathcal{R}^{m+1}_{++}) \times \ldots \times V^n(\mathcal{R}^{m+1}_{++}) \to \mathcal{R}$ is assumed to be continuous, increasing (Pareto-inclusive), and symmetric (anonymous). The Bergson-Samuelson indirect social-welfare function is $B\colon \mathcal{R}^{m+n}_{++} \to \mathcal{R}$ is given by

$$B(p,x) = W\big(V^1(p,x_1), \ldots, V^n(p,x_n)\big), \tag{1}$$

where $p = (p_1, \ldots, p_m)$ is the (common) price vector and $x = (x_1, \ldots, x_n)$ is the vector of individual expenditures. B is homogeneous of degree zero. Because W is increasing in individual utilities which are, in turn, increasing in individual expenditures, B is increasing in x.

Suppose prices change from p^b to p^a (b for 'before,' a for 'after'), and x gives the vector of individual expenditures corresponding to p^b (the 'before' expenditure vector). The common-scaling social cost-of-living index $I\colon \mathcal{R}^{2m+n}_{++} \to \mathcal{R}$ is written $I(p^a,p^b,x)$ and defined by

$$I(p^a,p^b,x) = \mu \Leftrightarrow W\big(V^1(p^b,x_1), \ldots, V^n(p^b,x_n)\big) = W\big(V^1(p^a,\mu x_1), \ldots, V^n(p^a,\mu x_n)\big), \tag{2}$$

or, equivalently, by

$$I(p^a, p^b, x) = \mu \Leftrightarrow B(p^b, x) = B(p^a, \mu x). \tag{3}$$

From the definition of I and the fact that B is increasing in x, $I(p^a, p^b, x) = 1$ if and only if $B(p^b, x) = B(p^a, x)$, $I(p^a, p^b, x) > 1$ if and only if $B(p^b, x) > B(p^a, x)$, and $I(p^a, p^b, x) < 1$ if and only if $B(p^b, x) < B(p^a, x)$ for all $(p^a, p^b, x) \in \mathcal{R}^{2m+n}_{++}$.

In this paper, we investigate the conditions needed to ensure that the index satisfies two standard index-number tests: the reversal and circular tests. Suppose that the index reports that prices have increased by 5% (index value = 1.05) when prices change from $\bar{p}$ to $\tilde{p}$ and 5% when prices change from $\tilde{p}$ to $\hat{p}$. Then, allowing for compounding, it is reasonable to expect the index to report an increase of 10.25% (index value = $(1.05)^2$) for a change from $\bar{p}$ to $\hat{p}$. This consistency condition is captured by the circular test. The reversal test, which is implied by the circular test, requires the index value for a change from p^b to p^a to be the reciprocal of the value for a change from p^a to p^b. We also investigate conditions needed to make the index independent of expenditures and consider the case of homothetic individual preferences as well as the general case. Finally, we ask whether the restriction for satisfaction of the tests is empirically reasonable.

In the general case, we demonstrate that the common-scaling index satisfies either test if and only if the Bergson-Samuelson indirect is homothetic in prices. This condition does not require homotheticity in expenditures, nor does it require individual preferences to be homothetic. When preferences are homothetic, however, satisfaction of the tests together with a preference-diversity condition does require the Bergson-Samuelson indirect to be homothetic in prices and homothetic in expenditures.

1. Homogeneity Properties of the Index

Because B is homogeneous of degree zero, the common-scaling index always satisfies several homogeneity properties.

Theorem 1: *The common-scaling cost-of-living index I is homogeneous of degree zero, homogeneous of degree one in its first argument, and jointly homogeneous of degree zero in its second and third arguments.*

Proof: For all $(p^a, p^b, x) \in \mathcal{R}^{2m+n}_{++}$, the definition of I implies

$$B(\lambda p^b, \lambda x) = B\big(\lambda p^a, I(\lambda p^a, \lambda p^b, \lambda x)\lambda x\big) \tag{4}$$

and, because B is homogeneous of degree zero,

$$B(p^b, x) = B\big(p^a, I(\lambda p^a, \lambda p^b, \lambda x)x\big). \tag{5}$$

From the definition of I,

$$I(\lambda p^a, \lambda p^b, \lambda x) = I(p^a, p^b, x), \tag{6}$$

and I is homogenous of degree zero.

Because B is homogeneous of degree zero,

$$B(p^b, x) = B\big(\lambda p^a, I(\lambda p^a, p^b, x)x\big) = B(p^a, [I(\lambda p^a, p^b, x)/\lambda]x). \tag{7}$$

By the definition of I,

$$I(\lambda p^a, p^b, x)/\lambda = I(p^a, p^b, x), \tag{8}$$

so

$$I(\lambda p^a, p^b, x) = \lambda I(p^a, p^b, x), \tag{9}$$

and the index is homogeneous of degree one in p^a.

By definition,

$$B(\lambda p^b, \lambda x) = B\big(\lambda p^a, I(\lambda p^a, \lambda p^b, \lambda x)\lambda x\big). \tag{10}$$

Because B is homogeneous of degree zero,

$$B(p^b, x) = B\big(\lambda p^a, I(\lambda p^a, \lambda p^b, \lambda x)\lambda x\big) = B\big(p^a, I(\lambda p^a, \lambda p^b, \lambda x)x\big). \tag{11}$$

Because I is homogenous of degree one in p^a and homogeneous of degree zero,

$$I(p^a, \lambda p^b, \lambda x) = \frac{1}{\lambda} I(\lambda p^a, \lambda p^b, \lambda x) = \frac{1}{\lambda} I(p^a, p^b, x), \tag{12}$$

and I is homogeneous of degree minus one in (p^b, x).

■

As might be expected, the common-scaling index is not, in general, homogeneous of degree minus one in p^b. The following section discusses the relationship between that property and satisfaction of the index-number tests.

2. The Reversal and Circular Tests

It is reasonable to require the value of the index for the change from p^b to p^a to be the reciprocal of the value for the reverse change. This condition is called the reversal test.

Reversal Test: A common-scaling index I satisfies the reversal test if and only if $I(p^a, p^b, x) = 1/I(p^b, p^a, x)$ for all $(p^a, p^b, x) \in \mathcal{R}^{2m+n}_{++}$.

Theorem 2 presents three conditions, each of which characterizes a common-scaling index that satisfies the reversal test. Two are conditions on the index, namely that it is homogeneous of degree minus one in its second argument or, equivalently, homogeneous of degree zero in its third argument. The third condition requires the Bergson-Samuelson indirect to be (conditionally) homothetic in prices.

Theorem 2: *If I is a common-scaling index,* *(a) I satisfies the reversal test if and only if it is homogeneous of degree minus one in its second argument;*

(b) I satisfies the reversal test if and only if it is homogeneous of degree zero in its third argument;

(c) I satisfies the reversal test if and only if the associated Bergson-Samuelson indirect B is homothetic in prices.

Proof: (1) First, we use Theorem 1 to show that I is homogeneous of degree minus one in its second argument if and only if it is homogeneous of degree zero in its third argument. If I is homogeneous of degree minus one p^b then, using Theorem 1,

$$I(p^a, p^b, \lambda x) = I(p^a/\lambda, p^b/\lambda, x) = \lambda I(p^a/\lambda, p^b, x) = I(p^a, p^b, x), \tag{13}$$

where the first equality follows from homogeneity of degree zero in all variables, the second follows from homogeneity of degree minus one in I's second argument, and the third follows from homogeneity of degree one in I's first argument.

If I is homogeneous of degree zero in x, then, using the result of Theorem 1,

$$I(p^a, \lambda p^b, x) = I(p^a/\lambda, p^b, x/\lambda) = \frac{1}{\lambda} I(p^a, p^b, x). \tag{14}$$

where the first inequality follows from I's homogeneity of degree zero in all its variables and the second inequality follows from I's homogeneity of degree one in its first argument and homogeneity of degree zero in x. Consequently I is homogeneous of degree minus one in its second argument.

(2) Next we show that, if I satisfies the reversal test, it is homogeneous of degree minus one in its second argument. Suppose I satisfies the reversal test. Then

$$I(p^a, p^b, x) = \frac{1}{I(p^b, p^a, x)} \tag{15}$$

and

$$I(p^a, \lambda p^b, x) = \frac{1}{I(\lambda p^b, p^a, x)} = \frac{1}{\lambda I(p^b, p^a, x)} = \frac{1}{\lambda} I(p^a, p^b, x), \tag{16}$$

and I is homogeneous of degree minus one in its second argument.

(3) The result of part (2) can be used to show that, if I satisfies the reversal test, B must be homothetic in prices. Suppose I satisfies the reversal test. From Theorem 1 and part (2), I must be homogeneous of degree zero in its first and second arguments. Then, for any $(\bar{p}, \hat{p}, x) \in \mathcal{R}^{2m+n}_{++}$ and any $\lambda \in \mathcal{R}_{++}$,

$$B(\bar{p}, x) = B(\hat{p}, x), \tag{17}$$

if and only if

$$I(\hat{p}, \bar{p}, x) = 1. \tag{18}$$

Because I is homogeneous of degree zero in its first two arguments

$$I(\lambda\hat{p}, \lambda\hat{p}, x) = 1, \tag{19}$$

so

$$B(\lambda\bar{p}, x) = B(\lambda\hat{p}, x) \tag{20}$$

and B is homothetic in prices.

(4) In this final part of the proof, we show that homotheticity of B in prices implies that I satisfies the reversal test. Suppose, therefore, that B is homothetic in prices. By the definition of I, for any $(\bar{p}, \hat{p}, x) \in \mathcal{R}_{++}^{2m+n}$ and any $\lambda \in \mathcal{R}_{++}$,

$$B\big(p^b, I(p^b, p^a, x)x\big) = B(p^a, x), \tag{21}$$

and

$$B\big(\lambda p^b, I(p^b, p^a, x)x\big) = B(\lambda p^a, x). \tag{22}$$

Setting $\lambda = I(p^b, p^a, x)$, homogeneity of degree zero of B implies

$$B(p^b, x) = B\big(I(p^b, p^a, x)p^a, x\big) = B\left(p^a, \frac{x}{I(p^b, p^a, x)}\right). \tag{23}$$

The definition of I requires

$$B(p^b, x) = B\big(p^a, I(p^a, p^b, x)x\big) \tag{24}$$

and, because (23) and (24) are true for any (p^a, p^b, x),

$$I(p^a, p^b, x) = \frac{1}{I(p^b, p^a, x)} \tag{25}$$

and the reversal test is satisfied.

■

The circular test requires the product of indexes for successive changes to be equal to the index for the complete change.

Circular Test: A common-scaling index I satisfies the circular test if and only if $I(\bar{p}, \tilde{p}, x)I(\tilde{p}, \hat{p}, x) = I(\bar{p}, \hat{p}, x)$ for all $(\bar{p}, \tilde{p}, \hat{p}, x) \in \mathcal{R}_{++}^{3m+n}$.

If $\bar{p}$ and $\hat{p}$ are both set equal to p^a and $\tilde{p}$ is set equal to p^b, the circular test's condition becomes the reversal test's condition. Consequently, satisfaction of the circular test implies satisfaction of the reversal test.

If a common-scaling index satisfies the reversal test, Theorem 3 demonstrates that it also satisfies the circular test.

Theorem 3: *I satisfies the reversal test if and only if it satisfies the circular test.*

Proof: Suppose I satisfies the reversal test and, for any $(\bar{p}, \tilde{p}, \hat{p}, x) \in \mathcal{R}^{3m+n}_{++}$, let $I(\bar{p}, \tilde{p}, x) = \bar{\mu}$ and $I(\tilde{p}, \hat{p}, x) = \hat{\mu}$. Then

$$B(\tilde{p}, x) = B(\bar{p}, \bar{\mu}x) \tag{26}$$

and

$$B(\hat{p}, x) = B(\tilde{p}, \hat{\mu}x). \tag{27}$$

Because B is homogeneous of degree zero, (26) implies

$$B(\tilde{p}, x) = B(\bar{p}/\bar{\mu}, x) \tag{28}$$

and (27) implies

$$B(\hat{p}, x) = B(\tilde{p}/\hat{\mu}, x). \tag{29}$$

Because I satisfies the reversal test, B is homothetic in p by Theorem 2, and, therefore, (28) implies

$$B(\tilde{p}/\hat{\mu}, x) = B(\bar{p}/\bar{\mu}\hat{\mu}, x), \tag{30}$$

and (28) and (29) imply

$$B(\hat{p}, x) = B(\bar{p}/\bar{\mu}\hat{\mu}, x). \tag{31}$$

Homogeneity of B implies

$$B(\hat{p}, x) = B(\bar{p}, \bar{\mu}\hat{\mu}x) \tag{32}$$

and, as a consequence, $I(\bar{p}, \hat{p}, x) = \bar{\mu}\hat{\mu} = I(\bar{p}, \tilde{p}, x)I(\tilde{p}, \hat{p}, x)$, and the circular test is satisfied.

Because satisfaction of the reversal test is implied by satisfaction of the circular test, if I satisfies the circular test, it satisfies the reversal test.

■

Homogeneity of degree zero of B together with homotheticity of B in p might be thought to imply homotheticity of B in x. That is the case if $n = 1$.

Theorem 4: *If $n = 1$, I satisfies the reversal test if and only if the individual's preferences are homothetic, with $V^1(p, x_1) = \phi^1\big(\alpha^1(p)x_1\big)$, where α^1 is homogeneous of degree minus one and ϕ is increasing.*

Proof: Suppose I satisfies the reversal test. Then B is homothetic in p and, because $B(p,x) = W\big(V^1(p,x_1)\big)$, V^1 must be homothetic in p. For any $(\bar{p},\hat{p},x_1)$,

$$V^1(\bar{p},x_1) = V^1(\hat{p},x_1) \Leftrightarrow V^1(\lambda\bar{p},x_1) = V^1(\lambda\hat{p},x_1) \Leftrightarrow V^1(\bar{p},x_1/\lambda) = V^1(\hat{p},x_1/\lambda) \qquad (33)$$

for any $\lambda > 0$, where the third equality follows from homogeneity of degree zero of V^1. Setting $\lambda = 1/x_1$, (33) becomes

$$V^1(\bar{p},x_1) = V^1(\hat{p},x_1) \Leftrightarrow V^1(\bar{p},1) = V^1(\hat{p},1). \qquad (34)$$

Because V^1 is homothetic in p, it is ordinally equivalent to the function α^1 which is homogeneous of degree minus one in p. It follows that

$$V^1(\bar{p},x_1) = V^1(\hat{p},x_1) \Leftrightarrow \alpha^1(\bar{p}) = \alpha^1(\hat{p}). \qquad (35)$$

Consequently, p is separable from x_1 in V^1, and

$$V^1(p,x_1) = \overset{*}{V}{}^1\big(\alpha^1(p),x_1\big) \qquad (36)$$

and, because V^1 is homogeneous of degree zero and α^1 is homogeneous of degree minus one,

$$V^1(p,x_1) = \overset{*}{V}{}^1\big(\alpha^1(p/x_1),1\big) = \overset{*}{V}{}^1\big(\alpha^1(p)x_1,1\big) = \phi^1\big(\alpha^1(p)x_1\big), \qquad (37)$$

where $\phi^1 = \overset{*}{V}{}^1(\cdot,1)$. Consequently, preferences are homothetic.

If preferences are homothetic, $V^1(p,x) = \phi^1\big(\alpha^1(p)x_1\big)$ where α^1 is homogeneous of degree one, so V^1 is homothetic in p. Consequently, B is homothetic in p and the reversal test is satisfied.

■

The result of Theorem 4 is not surprising if we realize that an individual index, defined analogously to the common-scaling index, is equal to the ratio of the expenditures needed to achieve the pre-change utility level of utility at p^a and p^b. To see this, note that the index for person 1 satisfies $I^1(p^b,p^a,x_1) = \nu$ if and only if $V^1(p^b,x_1) = V^1(p^a,\nu x_1) = u_1^b$. It follows that $\nu x_1 = E^1(u_1^b,p^a)$ where E^1 is the expenditure function that corresponds to V^1. Because $x_1 = E^1(p^b,x_1)$, $\nu = I^1(p^b,p^a,x^1) = E^1(u_1^b,p^a)/E^1(u_1^b,p^b)$. Thus Theorem 4 corresponds to the standard proofs that show that individual homotheticity is necessary and sufficient for satisfaction of the reversal and circular tests (Eichhorn [1978, ch. 3]).

If $n \geq 2$, B can be homothetic in p but not homothetic in x. Suppose, for example, that

$$V^i(p,x_i) = \begin{cases} \big(\alpha(p)x_i\big)^{\frac{1}{2}} & \text{if } i = 1; \\ \alpha(p)x_i & \text{if } i = 2,\ldots,n, \end{cases} \qquad (38)$$

where α is homogeneous of degree minus one, and

$$B(p,x) = \sum_{i=1}^{n} V^i(p,x_i) = \big(\alpha(p)x_1\big)^{\frac{1}{2}} + \sum_{i=2}^{n} \alpha(p)x_i. \qquad (39)$$

In this example, individual preferences are identical and homothetic. B is clearly not homothetic in x but it is homothetic in p. To see this, note that

$$B(\bar{p}, x) = B(\hat{p}, x) \Leftrightarrow \big(\alpha(\bar{p})x_1\big)^{\frac{1}{2}} + \sum_{i=2}^{n} \alpha(\bar{p})x_i = \big(\alpha(\hat{p})x_1\big)^{\frac{1}{2}} + \sum_{i=2}^{n} \alpha(\hat{p})x_i. \tag{40}$$

If $\alpha(\bar{p}) = \alpha(\hat{p})$, (40) is satisfied and, because $B(p, x)$ increases whenever $\alpha(p)$ increases, there are no other solutions. Because α is homogeneous of degree minus one,

$$\alpha(\bar{p}) = \alpha(\hat{p}) \Leftrightarrow \alpha(\lambda\bar{p}) = \alpha(\lambda\hat{p}) \tag{41}$$

for all $\lambda \in \mathcal{R}_{++}$, B is homothetic in p.

In the above example, individual preferences are homothetic and identical. In another example,

$$V^i(p, x_i) = \begin{cases} \frac{1}{\theta}\big(\alpha^i(p)x_i\big)^{\theta} + \beta^i(p); & \text{if } \theta \neq 0, \\ \ln\big(\alpha^i(p)x_i\big) + \beta^i(p); & \text{if } \theta = 0. \end{cases} \tag{42}$$

where α^i is homogeneous of degree minus one, $i \in \{1, \ldots, n\}$, $\theta \in \mathcal{R}$, and

$$B(p, x) = \sum_{i=1}^{n} V^i(p, x_i). \tag{43}$$

In this example, individual preferences may differ and are not homothetic unless $\beta^i(p)$ is independent of p. B is homothetic in p if and only if $\sum_{i=1}^{n} \beta^i(p)$ is independent of p. To see this, note that, if that condition obtains and $\theta \neq 0$,

$$B(\bar{p}, x) = B(\hat{p}, x) \Leftrightarrow \sum_{i=1}^{n} \big(\alpha^i(\bar{p})x_i\big)^{\theta} = \sum_{i=1}^{n} \big(\alpha^i(\hat{p})x_i\big)^{\theta}, \tag{44}$$

and

$$B(\lambda\bar{p}, x) = B(\lambda\hat{p}, x) \Leftrightarrow \sum_{i=1}^{n} \big(\alpha^i(\lambda\bar{p})x_i\big)^{\theta} = \sum_{i=1}^{n} \big(\alpha^i(\lambda\hat{p})x_i\big)^{\theta}. \tag{45}$$

Because each α^i is homogeneous of degree minus one, this is true if and only if

$$\lambda^{-\theta} \sum_{i=1}^{n} \big(\alpha^i(\bar{p})x_i\big)^{\theta} = \lambda^{-\theta} \sum_{i=1}^{n} \big(\alpha^i(\hat{p})x_i\big)^{\theta}, \tag{46}$$

$\lambda \in \mathcal{R}_{++}$, which implies $B(\bar{p}, x) = B(\hat{p}, x)$ so B is homothetic in p. A similar argument covers the case $\theta = 0$.

If $V^i(p, x_i) = \alpha(p)x_i + \beta^i(p)$, $i \in \{1, \ldots, n\}$, and

$$B(p, x) = \sum_{i=1}^{n} V^i(p, x_i) = \alpha(p) \sum_{i=1}^{n} x_i + \sum_{i=1}^{n} \beta^i(p), \tag{47}$$

individual cost-of-living indexes are given by

$$I^i(p^a, p^b, x_i) = \frac{\alpha(p^b)x_i + \beta^i(p^b) - \beta^i(p^a)}{\alpha(p^a)x_i}, \tag{48}$$

and the common-scaling index is

$$\frac{\alpha(p^b)\sum_{i=1}^n x_1 + \sum_{i=1}^n [\beta^i(p^b) - \beta^i(p^a)]}{\alpha(p^a)\sum_{i=1}^n x_i} = \sum_{i=1}^n \left[\frac{x_i}{\sum_{i=1}^n x_i}\right]\left[\frac{\alpha(p^b)x_i + \beta^i(p^b) - \beta^i(p^a)}{\alpha(p^a)x_i}\right], \tag{49}$$

which is the share-weighted sum of the individual indexes. Consequently, the plutocratic index is a special case of the common-scaling index, given appropriate social-evaluation and individual utility functions.[2] This index satisfies the reversal and circular tests if and only if $\sum_{i=1}^n \beta^i(p)$ is independent of p. In this formulation, individuals are identical at the margin: increases of decreases in expenditure change demands equally. This restriction (quasihomotheticity) is not supported by the data.

It might be interesting to know when the common-scaling index is independent of all individual expenditures. A characterization is provided by Theorem 5.

Theorem 5: *A common-scaling index I satisfies the reversal test and is independent of expenditures if and only if there exists a continuous function $\psi\colon \mathcal{R}_{++} \to \mathcal{R}$, homogeneous of degree one, and a continuous function $\tilde{B}\colon \psi(\mathcal{R}^m_{++}) \times \mathcal{R}^n_{++} \to \mathcal{R}$, decreasing in its first argument, increasing in x, and homogeneous of degree zero, such that B can be written as*

$$B(p, x) = \tilde{B}\big(\psi(p), x\big). \tag{50}$$

Proof: Suppose I satisfies the reversal test and is independent of x, so it can be written as $I(p^a, p^b, x) = \tilde{I}(p^a, p^b)$ for all $p^a, p^b \in \mathcal{R}^m_{++}$. By Theorem 3, I and $\tilde{I}$ satisfy the circular test so, for all $p^a, p^b, p \in \mathcal{R}^m_{++}$,

$$\tilde{I}(p^a, p^b) = \tilde{I}(p^a, p)\tilde{I}(p, p^b). \tag{51}$$

By the reversal test, $\tilde{I}(p, p^b) = 1/\tilde{I}(p^b, p)$. Setting $p = \mathbf{1}_m = (1, \ldots, 1) \in \mathcal{R}^m_{++}$,

$$\tilde{I}(p^a, p^b) = \frac{\tilde{I}(p^a, \mathbf{1}_m)}{\tilde{I}(p^b, \mathbf{1}_m)}. \tag{52}$$

Defining $\psi(\cdot) = \tilde{I}(\cdot, \mathbf{1}_m)$,

$$\tilde{I}(p^a, p^b) = \frac{\psi(p^a)}{\psi(p^b)}. \tag{53}$$

Because I and, therefore, $\tilde{I}$ are homogeneous of degree one in p^a, ψ is homogeneous of degree one.

[2] See Crossley and Pendakur [2009].

$I(p^a, p^b, x) = \mu$ if and only if

$$B(p^b, x) = B(p^a, \mu x). \tag{54}$$

Using (53) and homogeneity of B, this implies

$$B(p^b, x) = B\left(\frac{\psi(p^b)p^a}{\psi(p^a)}, x\right). \tag{55}$$

Setting $p^b = p$ and $p^a = \mathbf{1}_m$,

$$B(p, x) = B\left(\frac{\psi(p)\mathbf{1}_m}{\psi(\mathbf{1}_m)}, x\right). \tag{56}$$

Defining $\tilde{B}(\psi(p), x)$ by the right side of (56), (50) is satisfied. Because B is homogeneous of degree zero and ψ is homogeneous of degree one, $\tilde{B}$ is homogeneous of degree zero, and necessity is demonstrated.

To show sufficiency, suppose (50) is satisfied. Then

$$\tilde{B}\big(\psi(p^b), x\big) = \tilde{B}\big(\psi(p^a), \mu x\big) = \tilde{B}\left(\frac{\psi(p^a)}{\mu}, x\right), \tag{57}$$

for all $x \in \mathcal{R}^n_{++}$, where the last term follows from homogeneity of $\tilde{B}$. It follows that

$$I(p^a, p^b, x) = \mu = \frac{\psi(p^a)}{\psi(p^b)}, \tag{58}$$

which is independent of x.

■

In the example of (42) and (43), p is separable, so separability of p does not, in general, imply that B is homothetic in x.

The Pollak index satisfies the reversal and circular tests if the reference level of social welfare is constant. If, however, the index is computed using the before-change level of social welfare, the tests are not satisfied in general unless the index is independent of the level of social welfare (Blackorby and Donaldson [1983]). A similar observation applies to individual indexes defined as the ratio of expenditures needed to achieve a fixed utility level at prices p^a and p^b.

3. Individual Homotheticity

Individual price indexes are satisfy the reversal and circular tests if and only if preferences are homothetic. In that case, indirect utility functions can be written as

$$V^i(p, x_i) = \overset{*}{V}{}^i\left(\frac{x_i}{\pi^i(p)}\right) \text{ or } V^i(p, x_i) = \overset{*}{V}{}^i\big(\alpha^i(p)x_i\big), \tag{59}$$

where $\pi^i(p) = 1/\alpha^i(p)$, $\overset{*}{V}{}^i$, π^i and α^i are continuous, $\overset{*}{V}{}^i$ is increasing, π^i is homogeneous of degree one, and α^i is homogeneous of degree minus one. We assume, in addition, that there is a scaling of each π^i or, equivalently, α^i, such that $\overset{*}{V}{}^i$ is the same for all.[3] Thus

$$V^i(p, x_i) = \overset{*}{V}\left(\frac{x_i}{\pi^i(p)}\right) \text{ or } V^i(p, x_i) = \overset{*}{V}\big(\alpha^i(p)x_i\big). \tag{60}$$

The term $\alpha^i(p)y_i$ or, equivalently, $y_i/\pi^i(p)$ can be regarded as person i's real expenditure. If two people have the same real expenditures, (60) implies they are equally well off.

In the homothetic case, individual price indexes are

$$I^i(p^a, p^b, x_i) = \frac{\pi^i(p^a)}{\pi^i(p^b)} = \frac{\alpha^i(p^b)}{\alpha^i(p^a)}, \tag{61}$$

$i \in \{1, \ldots, n\}$.

The social-welfare function can be written as

$$B(p, x) = \bar{W}\left(\frac{x_1}{\pi^1(p)}, \ldots, \frac{x_n}{\pi^n(p)}\right) = \bar{W}\big(\alpha^1(p)x_1, \ldots, \alpha^n(p)x_n\big). \tag{62}$$

Symmetry of W and (60) imply symmetry of $\bar{W}$. The index $I(p^a, p^b, x)$ is equal to μ where

$$\bar{W}\big(a^1(p^b)x_1, \ldots, a^n(p^b)x_n\big) = \bar{W}\big(\alpha^1(p^a)\mu x_1, \ldots, \alpha^n(p^a)\mu x_n\big). \tag{63}$$

To discuss the consequences of the reversal and circular tests, let $y_i = Y^i(p, x_i) = \alpha^i(p)x_i$, $y = (y_1, \ldots, y_n) = Y(p, x) = (Y^1(p, x_1), \ldots, Y^n(p, x_n)) = (\alpha^1(p)x_1, \ldots, \alpha^n(p)x_n)$, and

$$\mathcal{Y}(x) = \big\{y = Y(p, x) \mid p \in \mathcal{R}_{++}\big\}. \tag{64}$$

y can be thought of as a vector of real expenditures, and $\mathcal{Y}(x)$ is the set of possible values for y given x. Because $a^i(p)\lambda x_i = a^i(p/\lambda)x_i$, $\mathcal{Y}(x)$ is a cone (if $y \in \mathcal{Y}(x)$, then $\lambda y \in \mathcal{Y}(x)$ for all positive λ). If all preferences are identical, $\mathcal{Y}(x)$ is a ray for each x. On the other hand, if $m \geq n$, and $\alpha^i(p) = 1/p_i$ for all i, $\mathcal{Y}(x) = \mathcal{R}^n_{++}$ for each x. It is always true that the union $\cup_{x \in \mathcal{R}^n_{++}} \mathcal{Y}(x) = \mathcal{R}^n_{++}$.

Reversibility has the consequence that, for each x, the restriction of $\bar{W}$ to $\mathcal{Y}(x)$ is homothetic.

Theorem 6: *Suppose individual preferences are homothetic and (60) holds. If a common-scaling index I satisfies the reversal test, then, for all $x \in \mathcal{R}^n_{++}$, $\bar{y}, \hat{y} \in \mathcal{Y}(x)$, and $\lambda \in \mathcal{R}_{++}$,*

$$\bar{W}(\bar{y}) = \bar{W}(\hat{y}) \Rightarrow \bar{W}(\lambda\bar{y}) = \bar{W}(\lambda\hat{y}). \tag{65}$$

[3] Scaling of these functions affects neither preferences nor individual cost-of-living indexes. When the aggregator functions $\overset{*}{V}{}^1, \ldots, \overset{*}{V}{}^n$ are the same, however, the choice is not arbitrary because $\alpha^1(p)x_1, \ldots, \alpha^n(p)x_n$ must be indexes of individual standards of living that can be compared interpersonally.

Proof: If $\bar{y}, \hat{y} \in \mathcal{Y}(x)$, there exist $\bar{p}, \hat{p} \in \mathcal{R}^m_{++}$ such that $\bar{y} = Y(\bar{p}, x)$ and $\hat{y} = Y(\hat{p}, x)$. By Theorem 2, satisfaction of the reversal test implies that B is homothetic in p. Consequently,

$$\bar{W}\big(Y(\bar{p}, x)\big) = \bar{W}\big(Y(\hat{p}, x) \Leftrightarrow \bar{W}\big(Y(\bar{p}/\lambda, x) = \bar{W}\big(Y(\hat{p}/\lambda, x)\big) \tag{66}$$

for all $\lambda \in \mathcal{R}_{++}$. Because $Y(p/\lambda, x) = \lambda Y(p, x)$, for all (p, x),

$$\bar{W}(\bar{y}) = \bar{W}(\hat{y}) \Leftrightarrow \bar{W}\big(\lambda Y(\bar{p}, x)\big) = \bar{W}\big(\lambda Y(\hat{p}, x)\big) \Leftrightarrow \bar{W}(\lambda \bar{y}) = \bar{W}(\lambda \hat{y}) \tag{67}$$

for all $\lambda \in \mathcal{R}_{++}$.

■

By itself, Theorem 6 does not imply that $\bar{W}$ is homothetic. Suppose individual utility functions satisfy $\alpha^i(p) = \alpha(p)$, $i = 1, \ldots, n$, so that preferences are identical and homothetic, with

$$\bar{W}(y) = y_1^{1/2} + \sum_{i=2}^{n} y_i. \tag{68}$$

The set $\mathcal{Y}(x)$ is the set of all $y = \big(\alpha(p)x_1, \ldots, \alpha(p)x_n\big)$, which is a set of scalar multiples of the vector x and is, therefore, a ray. In that case, the common-scaling index is the common individual index, the property of equation (65) is trivially satisfied, but $\bar{W}$ is not homothetic.

To show that satisfaction of the reversal test implies that $\bar{W}$ is homothetic, a preference diversity condition is needed. In the definition, an ϵ-neighbourhood of y is $\{z \mid ||z - x|| < \epsilon\}$, $\epsilon \in \mathcal{R}_{++}$.

Preference Diversity:

(a) For all $p, x \in \mathcal{R}^{m+n}_{++}$, $\mathcal{Y}(x)$ contains an ϵ-neighbourhood of $Y(p, x)$ for some $\epsilon \in \mathcal{R}_{++}$;

(b) For all $\bar{y}, \hat{y} \in \mathcal{R}^n_{++}$, there exist $x^1, \ldots, x^R \in \mathcal{R}^n_{++}$ and $y^1, \ldots, y^{R+1} \in \mathcal{R}^n_{++}$, such that

$$y^1 = \bar{y} \in \mathcal{Y}(x^1) \tag{69}$$

$$y^r \in \mathcal{Y}(x^{r-1}) \cap \mathcal{Y}(x^r), r \in \{2, \ldots, R\}, \tag{70}$$

and

$$y^{R+1} = \hat{y} \in \mathcal{Y}(x^R). \tag{71}$$

Part (a) of preference diversity implies that the sets $\mathcal{Y}(x)$ are thick. This means that changes in prices alone are capable of moving y in any direction. For that reason, it implies that the the number of goods is no smaller than the number of people ($m \geq n$). Part (b) ensures that any two values of y can be connected with overlapping sets $\mathcal{Y}(x^1), \ldots, \mathcal{Y}(x^R)$. Because the xs and ys in Part (b) are not required to be distinct, the case in which there exists $x \in \mathcal{R}^n_{++}$ such that $\bar{y}, \hat{y} \in Y(x)$ is covered. It may be true that one part of the axiom implies the other but we have been unable to prove it.

Part (b) of preference diversity is sufficient to extend to result of Theorem 6 to all of $\mathcal{R}^n_{++}$.

Theorem 7: *Suppose individual preferences are homothetic, satisfy preference diversity, and (60) holds. Then a common-scaling index I satisfies the reversal test if and only if $\bar{W}$ is homothetic.*

Proof: Suppose I satisfies preference diversity and the reversal test. And consider any $\bar{y}, \hat{y} \in \mathcal{R}^n_{++}$ with $\bar{W}(\bar{y}) = \bar{W}(\hat{y})$. Then, using part (b) of preference diversity, define $\tilde{y}^r = \tilde{\lambda} y^r$ such that $\bar{W}(\tilde{y}^r) = \bar{W}(\hat{y})$, $r = 2, \ldots, R$. Because $\mathcal{Y}(x)$ is a cone for each x, $\tilde{y}^r \in \mathcal{Y}(x^r) \cap \mathcal{Y}(x^{r+1})$, $r \in \{1, \ldots, R-1\}$. Repeated application of the result of Theorem 6 yields

$$\bar{W}(\lambda \bar{y}) = \bar{W}(\lambda \tilde{y}^2) \tag{72}$$

for all $\lambda \in \mathcal{R}_{++}$, and

$$\bar{W}(\lambda \tilde{y}^r) = \bar{W}(\lambda \tilde{y}^{r+1}), \tag{73}$$

$r = 1, \ldots, R$. (72) and (73) together imply

$$\bar{W}(\lambda \bar{y}) = \bar{W}(\lambda \hat{y}) \tag{74}$$

for all $\lambda \in \mathcal{R}_{++}$.

Sufficiency is immediate.

■

Any homothetic $\bar{W}$, including the members of the S-Gini family of social-welfare functions, is sufficient.[4] If $\bar{W}$ is homothetic, B must be separately homothetic in p and x. This observation is stated in a corollary to Theorem 7.

Corollary 7: *Suppose individual preferences are homothetic and satisfy preference diversity. Then a common-scaling index I satisfies the reversal test if and only if the Bergson-Samuelson indirect B is homothetic in prices and homothetic in expenditures.*

If $\bar{W}$ is homothetic, the index takes on a simple form. The equally-distributed equivalent real expenditure ξ (EDERE) for real-expenditure vector y is that level of expenditure which, if experienced by everyone, is as good as the actual distribution. Thus, $\xi = \Xi(y)$, where $\Xi\colon \mathcal{R}^n_{++} \to \mathcal{R}$ is defined by

$$\bar{W}(\xi \mathbf{1}_n) = \bar{W}(y). \tag{75}$$

Ξ is ordinally equivalent to $\bar{W}$ and, because $\bar{W}$ is homothetic, Ξ is homogeneous of degree one.[5] In addition, if every person has the same real expenditure, the EDERE is equal to it. Let $y^b = Y(p^b, x)$ and $y^a = Y(p^a, x)$. Because Ξ is ordinally equivalent to $\bar{W}$, $I(p^a, p^b, x) = \mu$ if and only if $\Xi(y^b) = \Xi(\mu y^a)$. Because Ξ is homogeneous of degree one,

$$I(p^a, p^b, x) = \frac{\Xi(y^b)}{\Xi(y^a)}. \tag{76}$$

[4] See Donaldson and Weymark [1980].

[5] Let $\xi' = \Xi(\lambda y)$, $\lambda \in \mathcal{R}_{++}$. Then $\bar{W}(\xi' \mathbf{1}_n) = \bar{W}(\lambda y)$. Because $\bar{W}$ is homothetic, this implies $\bar{W}\big((\xi'/\lambda)\mathbf{1}_n\big) = \Xi(y)$, and $\xi' = \Xi(\lambda y) = \lambda \Xi(y)$.

Thus the index measures the percentage gain or loss in EDERE. The Atkinson [1970]-Kolm [1969]-Sen [1973] index of real-expenditure inequality τ is given by $T\colon \mathcal{R}^n_{++} \to \mathcal{R}$, where

$$\tau = T(y) = \frac{M(y) - \Xi(y)}{M(y)} \tag{77}$$

where $M(y)$ is the mean of the elements of y.[6] Homogeneity of degree one of Ξ implies that T is homogeneous of degree zero, so it measures relative inequality. (76) can be rewritten as

$$I(p^a, p^b, x) = \frac{M(y^b)\big[1 - T(y^b)\big]}{M(y^a)\big[1 - T(y^a)\big]}. \tag{78}$$

The index is positive (negative) if a price change decreases (increases) mean real expenditure, inequality constant, or if a change increases (decreases) inequality, mean real expenditure constant.

Theorem 8 finds the conditions that make the index independent of all expenditures. In the proof, the unit vector $e \in \mathcal{E} = \{e \in \mathcal{R}^n_{++} \mid ||e|| = 1\}$, and e^i is a unit vector with $e^i_i = 1$ and $e^i_k = 0$, $k \neq i$.

Theorem 8: *Suppose individual preferences are homothetic, satisfy preference diversity, and (60) holds. A common-scaling index satisfies the reversal test and is independent of expenditures if and only if*

$$\Xi(y) = \prod_{i=1}^n {y_i}^{\frac{1}{n}}, \tag{79}$$

and

$$I(p^a, p^b, x) = \prod_{i=1}^n \left(\frac{\pi^i(p^a)}{\pi^i(p^b)}\right)^{\frac{1}{n}} = \prod_{i=1}^n \left(\frac{\alpha^i(p^b)}{\alpha^i(p^a)}\right)^{\frac{1}{n}} = \prod_{i=1}^n I^i(p^a, p^b)^{\frac{1}{n}}. \tag{80}$$

Proof: By Theorem 7, the reversal test and preference diversity imply (76). If I is independent of x,

$$I(p^a, p^b, x) = \frac{\Xi(y^b)}{\Xi(y^a)} = \frac{\Xi\big(\alpha^1(p^b), \ldots, \alpha^n(p^b)\big)}{\Xi\big(\alpha^1(p^a), \ldots, \alpha^n(p^a)\big)} = \frac{\phi(p^b)}{\phi(p^a)}, \tag{81}$$

where $\phi(p) = \Xi\big(\alpha^1(p), \ldots, \alpha^n(p)\big)$ for all p. It follows that

$$\Xi(y) = \Xi\big(\alpha^1(p)x_1, \ldots, \alpha^n(p)x_n\big) = \phi(p)f(x) \tag{82}$$

for some function f. Defining $z_i = \alpha^i(x)$, $i = 1, \ldots, n$, (82) becomes

$$\Xi(z_1x_1, \ldots, z_nx_n) = \phi(p)f(x) = \bar{\phi}(z)f(x), \tag{83}$$

[6] For a survey of ethical indexes of inequality, see Blackorby, Bossert and Donaldson [1999].

where $\bar{\phi}(z) = \bar{\phi}\big(\alpha^1(p), \dots, \alpha^n(p)\big) = \phi(p)$. Part (a) of preference diversity implies that, $\bar{y} = \bar{z}\bar{x}$ can be moved through an ϵ-neighbourhood by changing p, which changes z, $\bar{x}$ constant. Each element of the set

$$\big\{z \mid ||(z_1\bar{x}_1, \dots, z_n\bar{x}_n) - (\bar{z}_1\bar{x}_1, \dots, \bar{z}_n\bar{x}_n)|| < \epsilon\big\} \tag{84}$$

or

$$\Big\{z \mid ||\big((z_1 - \bar{z}_1)\bar{x}_1, \dots, (z_n - \bar{z}_n)\bar{x}_n\big)|| < \epsilon\Big\} \tag{85}$$

is a feasible value of z, $x = \bar{x}$. If z is restricted to move in direction $e \in \mathcal{E}$ only, then, writing $z = \bar{z} + \delta e$, each element of the set

$$\big\{\delta \mid ||\big(\delta e_1\bar{x}_1, \dots, \delta e_n\bar{x}_n\big)|| < \epsilon\big\} \tag{86}$$

is a feasible value for δ. Because $y = \bar{z}\bar{x}$ if and only if $\delta = 0$, the possible values for δ comprise an open interval which includes zero. If, for example, $e = e^i$, the set of possible values for δ is the interval $(-\epsilon/\bar{x}_i, \epsilon/\bar{x}_i)$. Locally, therefore, each element of z can be moved independently. It follows that, in an ϵ-neighbourhood of each $y \in \mathcal{R}^n_{++}$, (83) is a Pexider equation whose solution is (Eichhorn [1978, Theorem 3.6.4, p. 67])

$$\Xi(y) = \Xi\big((z_1x_1, \dots, z_nx_n)\big) = \bar{c}\prod_{i=1}^{n} {y_i}^{k_i}, \tag{87}$$

$\bar{c} > 0$. Because Ξ is symmetric and increasing, the k_is must be equal and and positive. Homogeneity of degree one of Ξ further requires $k_i = 1/n$, $i = 1, \dots, n$. Because $\Xi(\gamma\mathbf{1}_n) = \gamma$ for all $\gamma \in \mathcal{R}_{++}$, $\bar{c} = 1$. yielding (79) which, in turn, implies (80). Because x can take on any value in $\mathcal{R}^n_{++}$, the ϵ-neighbourhoods overlap, and (87) can be extended to all of $\mathcal{R}^n_{++}$. Sufficiency is immediate.

■

The Cobb-Douglas indexes are not the only reasonable indexes. There are good reasons to suppose that the social-welfare function is additively separable.[7] If Ξ is continuous, increasing, homogeneous of degree one and additively separable, it must satisfy[8]

$$\Xi(y) = \begin{cases} \left[\frac{1}{n}\sum_{i=1}^n {y_i}^r\right]^{\frac{1}{r}} & \text{if } r \in \mathcal{R}\backslash\{0\}; \\ \prod_{i=1}^n y_i^{\frac{1}{n}} & \text{if } r = 0. \end{cases} \tag{88}$$

The function exhibits inequality aversion in real expenditures if and only if the parameter $r < 1$. The index is easy to compute using (76), with

$$I(p^a, p^b, x) = \begin{cases} \dfrac{\left[\sum_{i=1}^n (y_i^b)^r\right]^{\frac{1}{r}}}{\left[\sum_{i=1}^n (y_i^a)^r\right]^{\frac{1}{r}}} = \dfrac{\left[\sum_{i=1}^n \left(\alpha^i(p^b)x_i\right)^r\right]^{\frac{1}{r}}}{\left[\sum_{i=1}^n \left(\alpha^i(p^a)x_i\right)^r\right]^{\frac{1}{r}}} & \text{if } r \in \mathcal{R}\backslash\{0\}; \\ \prod_{i=1}^n \left[\dfrac{\alpha^i(p^b)}{\alpha^i(p^a)}\right]^{\frac{1}{n}} & \text{if } r = 0. \end{cases} \tag{89}$$

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[7] See Blackorby, Bossert and Donaldson [2005, ch. 4].

[8] See Eichhorn [1978, ch. 2] for a proof.

Blackorby and Donaldson [1983] solved a very different problem, but their indexes are related to the ones found in this article. One of their cases is the Cobb-Douglas index of Theorem 8. Another is related to the mean-of-order r indexes $r \neq 0$. but lacks the expenditure terms. This occurs because of the minimizing implicit in Pollak's index.

4. Approximation of the Common-Scaling Index

The individual cost-of-living index satisfies the reversal test if and only if individual preferences are homothetic. Theorem 2 shows that if the common-scaling index satisfies the reversal test if and only if B is homothetic in p. Equivalently, the common-scaling index I satisfies the reversal test if and only if I is homogeneous of degree minus one in p^b (it is homogeneous of degree one in p^a regardless of whether it satisfies the reversal test). Homotheticity of B in p does not require individual preferences to be homothetic. In this section, we consider how to approximate I when when B is homothetic in p. In addition, we consider the empirical magnitude of these restrictions in a Monte Carlo experiment.

Define the normalized proportionate welfare-weight function, $\phi_i(p,x)$, as

$$\phi_i(p,x) = \frac{\nabla_{\ln x_i} B(p,x)}{\sum_{j=1}^{n} \nabla_{\ln x_j} B(p,x)}, \tag{90}$$

and let normalized *reference* proportionate welfare weights, $\overline{\phi}_i$, be those evaluated at p^b (so that $\overline{\phi}_i = \phi_i(p^b, x)$). Let $w = (w_1, ..., w_m)$ be the $m-$vector of budget shares where w_j is defined as the share of total expenditure commanded by the jth commodity. Let w^i be the budget-share vector of the ith household, and let $\overline{w}^i = w^i(p^b, x_i)$ be the value of the budget share vector for household i facing 'before' prices p^b and expenditure x_i. The following proposition, reproduced from Crossley and Pendakur [2009], establishes the second-order approximation of the common-scaling index. In the theorem statement, tildes (with no subscript i, and no superscript) denote $\overline{\phi}_i$-weighted averages.

Theorem 9: *Given the normalized reference proportionate welfare weights, $\overline{\phi}_i$, their local price responses, $\nabla_{\ln p'}\phi_i\left(p^b, x\right)$, reference budget shares, $\overline{w}^i$, and their price and expenditure derivatives, the second-order approximation of the common-scaling index Π^* is*

$$\ln \Pi^* \approx dp'\widetilde{w} + \frac{1}{2}dp'\left[\widetilde{w}\widetilde{w}' - diag\left(\widetilde{w}\right) + \widetilde{\nabla_{\ln p'} w} + \widetilde{\nabla_{\ln x} w \widetilde{w}'} + \widetilde{\Phi}\right] dp \tag{91}$$

where

$$dp = \ln p^a - \ln p^b, \tag{92}$$

$$\widetilde{w} \equiv \sum_{i=1}^{N} \overline{\phi}_i \overline{w}^i, \tag{93}$$

$$\widetilde{\nabla_{\ln p'} w} \equiv \sum_{i=1}^{N} \overline{\phi}_i \nabla_{\ln p'} w^i \left(p^b, x_i\right), \tag{94}$$

$$\widetilde{\nabla_{\ln x} w} = \sum_{i=1}^{N} \overline{\phi}_i \nabla_{\ln x} w^i(p^b, x_i), \tag{95}$$

and

$$\widetilde{\Phi} \equiv \sum_{i=1}^{N} {}_i\overline{w}^i \nabla_{\ln p'} \phi_i \left(p^b, x\right). \tag{96}$$

Consider first the restriction that the common-scaling index is homogeneous of degree -1 in p^b. Note that the first-order part of the approximation, $dp'\widetilde{w}$, satisfies this homogeneity by construction. Given, this, the second-order part of the approximation must be homogeneous of degree 0 in p^b in order for the approximation as a whole to be homogeneous of degree 1 in p^b. Homogeneity of degree zero is equivalent to

$$\begin{aligned}&\left(\lambda + \ln p^a - \ln p^b\right)' \left[\widetilde{w}\ \widetilde{w}' - diag\left(\widetilde{w}\right) + \widetilde{\nabla_{\ln p'} w} + \widetilde{\nabla_{\ln x} w}\ \widetilde{w}' + \widetilde{\Phi}\right] \left(\lambda + \ln p^a - \ln p^b\right) \\ &= \left(\ln p^a - \ln p^b\right)' \left[\widetilde{w}\ \widetilde{w}' - diag\left(\widetilde{w}\right) + \widetilde{\nabla_{\ln p'} w} + \widetilde{\nabla_{\ln x} w}\ \widetilde{w}' + \widetilde{\Phi}\right] \left(\ln p^a - \ln p^b\right).\end{aligned} \tag{97}$$

This holds if and only if the row and column sums of the matrix in the square brackets are zero. Note that $[\widetilde{w}\ \widetilde{w}' - diag\left(\widetilde{w}\right)]$ have row and column sums of zero due to the fact that budget shares sum to 1. Note also that since the budget-share vector everywhere sums to 1, the sum of its derivatives with respect to any single argument over the m budget-shares is zero, so that $\lambda'\widetilde{\nabla_{\ln p'} w} = 0_m'$ and $\lambda'\widetilde{\nabla_{\ln x} w} = 0$. Further, since individual preferences satisfy homogeneity in p, x, $\widetilde{\nabla_{\ln p'} w}\lambda + \widetilde{\nabla_{\ln x} w}\ \lambda = 0_m$. Consequently, $\left[\widetilde{w}\ \widetilde{w}' - diag\left(\widetilde{w}\right) + \widetilde{\nabla_{\ln p'} w} + \widetilde{\nabla_{\ln x} w}\ \widetilde{w}' + \widetilde{\Phi}\right]$ has row and column sums of zero if and only if $\widetilde{\Phi}$ has row and column-sums of zero.

Since $\overline{w}^i$ sums to 1, and $\widetilde{\Phi} \equiv \sum_{i=1}^{N} \overline{w}^i \nabla_{\ln p'} \phi_i(p^b, x)$, the restriction is satisfied if and only if $\nabla_{\ln p'} \phi_i(p^b, x) = 0_m$. In this case, $\widetilde{\Phi} = 0$, and the second-order approximation satisfying homogeneity of degree -1 in p^b is given by

$$\Pi^* \approx 1 + dp'\widetilde{w} + \frac{1}{2} dp' \left[\widetilde{w}\ \widetilde{w}' - diag\left(\widetilde{w}\right) + \widetilde{\nabla_{\ln p'} w} + \widetilde{\nabla_{\ln x} w}\ \widetilde{w}'\right] dp. \tag{98}$$

Consider next the restriction that individual utility functions are homothetic in p. In this case, it is well known that all individuals would have $\nabla_{\ln x} w^i(p^b, x_i) = 0$, and thus

$$\widetilde{\nabla_{\ln x} w} = 0, \tag{99}$$

leading to a second order approximation given by

$$\Pi^* \approx 1 + dp'\widetilde{w} + \frac{1}{2} dp' \left[\widetilde{w}\ \widetilde{w}' - diag\left(\widetilde{w}\right) + \widetilde{\nabla_{\ln p'} w}\right] dp. \tag{100}$$

To sum up, if we restrict $\widetilde{\Phi} = 0$, then the second-order approximation of the common-scaling index will satisfy the reversibility test, and if we further restrict $\widetilde{\nabla_{\ln x} w} = 0$, then each individual cost-of-living index will satisfy the reversibility test. Now we consider the empirical magnitude of these restrictions.

5. Monte Carlo Experiment

Here, we set up an experiment to establish the relative size of these two restrictions in a plausible economic environment. We consider the same environment as Crossley and Pendakur (2009), which uses the data and empirical specification developed in Pendakur (2002). We consider an environment with preference heterogeneity, driven by a single demographic characteristic z. To mimic the data used by Pendakur (2002), we use a single demographic characteristic, the number of household members, and draw 1000 values of log-expenditure and the number of household members from independent standard normals with means of 4.54 and 2.17, respectively, and standard deviations of 0.65 and 1.31, respectively. The number of household members is discretized to $1, 2, 3 \ldots$.

For consumer preferences, we use the 9-good parametric demand system estimated by Pendakur (2002) in which all households have single demographic characteristic, z, equal to the number of household members and have Quadratic Almost Ideal (QAI) indirect utility (see Banks, Blundell and Lewbel 1997) given by:

$$V(p, x, z) = h\left(\frac{\ln x - \ln a(p, z)}{b(p) + q(p)\left(\ln x - \ln a(p, z)\right)}\right) \tag{101}$$

where h is monotonically increasing. Here, we use the following functional forms for a, b, and q:

$$\ln a(p, z) = d_0 + dz + \ln p'a + \ln p'Dz + \frac{1}{2}\ln p'A_0 \ln p + \frac{1}{2}\ln p'A_z \ln pz, \tag{102}$$

$$\ln b(p) = \ln p'b, \tag{103}$$

and

$$q(p) = \ln p'q, \tag{104}$$

where all parameter values are taken from Pendakur (2002, Table 3) and $\iota'a = 1$, $\iota'b = \iota'q = 0$, $\iota'D = \iota'A_0 = \iota'A_z = 0_m$, $A_0 = A_0'$ and $A_z = A_z'$. For convenience, let the reference price vector be $p^b = 1_m \cdot 100$ and let $d_0 = -\ln 100$. Thus, $\ln a(p^b, z) = \ln b(p^b) = q(p^b) = 0$.

In this model, preference heterogeneity is entirely captured by z. Thus, we can think of this utility function as of the following form:

$$V^i(p, x_i) = h\left(\frac{\ln x - \ln a^i(p)}{b(p) + q(p)\left(\ln x - \ln a^i(p)\right)}\right). \tag{105}$$

If $b(p) = q(p) = 0$, then preferences are homothetic and heterogeneous across individuals, and fall into the class studied in Section 3 and given by equation (48). If $q(p) = 0$ but $b(p) \neq 0$, then preferences fall into Deaton and Muelbauer's Almost Ideal class (logarithmic Gorman

Polar form) and have preference heterogeneity coming through level effects in budget shares. If $q(p) \neq 0$, then budget shares are quadratic in the log of expenditure and fall into the Quadratic Almost Ideal class of Banks, Blundell and Lewbel (1997).

For this indirect utility function, reference utility is

$$\overline{u}_i = V(p^b, x_i, z_i) = h(\ln x_i - dz_i). \tag{106}$$

Consequently, the function h determines the marginal utility of money. If $h(t) = t$, then $V(p^b, x, z) = \ln x - dz_i$ and, at a vector of unit prices, the marginal utility of a proportionate increase in money is constant. If, in contrast, $h(t) = \exp(t)$, then $V(p^b, x, z) = x/\exp(dz_i)$, so that at a vector of unit prices, the marginal utility of money is constant $(1/\exp(dz_i))$, and the marginal utility of a proportionate increase in money is proportional to expenditure $(x/\exp(dz_i))$. Since h is a monotonic transformation of utility, it drops out of all calculations of behavioural responses, but structure on h is required to compute welfare weights.

For the social welfare function, we use the generalized utilitarian form with $W(u_1, ..., u_N) = \sum_{i=1}^{N} g(u_i)$. This yields utilitarianism, which is neutral to inequality of utilities, if $g(u_i) = u_i$. It yields social welfare which is somewhat averse to inequality in utilities if $g(u_i) = \ln u_i$ and strongly averse if $g(u_i) = -(u_i)^{-1}$. These welfare functions correspond to the *Mean-of-Order r* or *Atkinson,* class of social welfare functions with social welfare ordinally equivalent to the arithmetic mean of utility (if $g(u_i) = u_i$), the geometric mean of utility (if $g(u_i) = \ln u_i$) or the harmonic mean of utility (if $g(u_i) = -(u_i)^{-1}$). Obviously, the indirect welfare function, B, is the compound function $W\left(V^1(p, x_1), ..., V^n(p, x_n)\right)$.

Now, we need to evaluate the sizes of $\widetilde{\nabla_{\ln x} w \; w'}$ and $\widetilde{\Phi}$ in the context of the second-order approximation. Let $\widetilde{S}$ denote the second order part of the approximation, excluding $\widetilde{\Phi}$:

$$\widetilde{S} = \widetilde{w}\ \widetilde{w}' - diag(\widetilde{w}) + \widetilde{\nabla_{\ln p'} w} + \widetilde{\nabla_{\ln x} w\ w'} \tag{107}$$

Since the second-order approximation of the common-scaling index uses quadratic forms, we compare the sizes of $dp'\widetilde{\nabla_{\ln x} w\ w'}dp$ and $dp'\widetilde{\Phi}dp$ to $dp'\widetilde{\mathbf{S}}dp$ over many random draws of dp. We draw 1000 observations of the vector dp from independent random uniform distributions, each with minimum -0.45 and maximum 0.55. The uniform distribution chosen has small but positive mean in order to reflect underlying price growth. The independence of prices from each other allows for lots of relative price variation. For each draw, dp^t, $t = 1, ..., 1000$, we compute the ratios $\left| dp^{t\prime}\widetilde{\Phi}dp^t dp^{t\prime}\widetilde{\mathbf{S}}dp^t \right|$ and $\left| \frac{dp^{t\prime}\widetilde{\nabla_{\ln x} w\ w'}dp^t}{dp^{t\prime}\widetilde{\mathbf{S}}dp^t} \right|$. We use the absolute value to account for the fact that this ratio may be positive or negative for different price vectors, because the numerator and denominator need not have the same sign. Table 1 presents the average over the 1000 draws of $\left| dp^{t\prime}\widetilde{\Phi}dp^t / dp^{t\prime}\widetilde{\mathbf{S}}dp^t \right|$ for cases with varying marginal utility, social welfare functions and preference structures. (Recall that QAI demands have quadratic Engel curves and AI demands have linear Engel curves.) Note that the random number generator seed is held constant across these experiments, so that the randomly generated population $\{x_i, \mathbf{z}_i\}_{i=1}^{n}$ is identical across rows in the table.

The main lessons from Table 1 are that: (1) $\widetilde{\Phi}$ is smaller than $\widetilde{\nabla_{\ln x}w \; \widetilde{w}'}$ under plausible circumstances; and (2) although the non-homotheticity of B, captured by $\widetilde{\Phi}$, may be thought of as so small that it may be ignored, the non-homotheticity of V, captured by $\widetilde{\nabla_{\ln x}w \; \widetilde{w}'}$, is probably not so small that it can be ignored.

The average of the 1000 values of the $\left|dp^{t\prime}\widetilde{\Phi}dp^{t}/dp^{t\prime}\widetilde{\mathbf{S}}dp^{t}\right|$ takes its largest value of 0.084 for the common-scaling index with $g(u_i) = \ln u_i$ and $h(t) = \exp(t)$ combined with QAI preferences. This suggests that even in this case the term driven by $\widetilde{\Phi}$ is at least an order of magnitude smaller than the second-order part of the approximation as a whole. Indeed, $\widetilde{\Phi}$ can be exactly zero under plausible circumstances. When utility is Almost Ideal and marginal utility is log-money metric and welfare is utilitarian, $\widetilde{\Phi}$ is exactly zero. When $g(u_i) = u_i$, $h(t) = \exp(t)$ and preferences are QAI, the average value of $\left|dp_t'\widetilde{\Phi}dp_t/dp_t'\widetilde{\mathbf{S}}dp_t\right|$ is less than $1/20th$. So, imposing reversibility in the approximation of the common-scaling index by ignoring the trailing term $\widetilde{\Phi}$ may not be too damaging in empirical work.

The term capturing non-homotheticities of individual preferences, $E\left[\left|\frac{dp'\widetilde{\nabla_{\ln x}w \; \widetilde{w}'}dp}{dp'\widetilde{\mathbf{S}}dp}\right|\right]$, is based on the income effects in budget share equations, $\widetilde{\nabla_{\ln x}w}$. Ignoring this is more costly. Between one-tenth and one-half of the second-order part of the approximation is driven by the value of $\widetilde{\nabla_{\ln x}w \; \widetilde{w}}$. Thus, we conclude that it is prudent to include this term in empirical work.

The bottom line here is as follows. A huge body of evidence tells us that individual preferences do not satisfy homotheticity, so that individual cost-of-living indices should not satisfy the reversal test. The common-scaling index also satisfies the reversal test if individual preferences satisfy homotheticity. However, restricting the common-scaling index to this case imposes large costs on the accuracy of the empirical estimates. In the theoretical part of this paper, we show that the common-scaling index satisfies the reversal test if and only if a much weaker condition, homotheticity of the indirect welfare function, holds. Restricting the common-scaling index to this case imposes some cost on the accuracy of empirical estimates, but, in our view, this loss of accuracy is small relative to the gain of having a social cost-of-living index that satisfies the reversal and circular tests.

6. Concluding Remarks

In this note, we demonstrate that the common-scaling social cost-of-living index satisfies the reversal or circular tests if and only if the Bergson-Samuelson indirect social-welfare function B is homothetic in prices, which provides a simple test for estimation. Equivalently, the index satisfies either test if and only if it is homogeneous of degree minus one in its second argument. This condition does not require homotheticity of individual preferences and accommodates a wide range of social-welfare functions.

There are cases in which the index is independent of individual expenditures. That property is satisfied if and only if prices are separable in B.

If individual preferences are homothetic, a preference diversity axiom is satisfied, and the social-welfare function is additively separable (in utilities), the index can be written as in (89), which is the ratio of mean-of-order-r equally distributed real expenditures before and after the price change. If parameter r is less than one, the social-evaluation function exhibits aversion to inequality of real expenditures. And, if r is zero, the common-scaling index is independent of individual expenditures. We know, however, that individual preferences are not homothetic, and this suggests that the more general formulation should be used if resources for estimation are available. The social-welfare function need not be additively separable as long as the Bergson-Samuelson indirect is homothetic in prices *and* homothetic in expenditures.

In order to satisfy the tests, the indirect Bergson-Samuelson social-welfare function must be homothetic in prices, a restriction which affects estimation. Because available data are attached to households rather than individuals, estimation should be able to incorporate the fact of diverse household types. Crossley and Pendakur [2009] employ equivalence scales to deal with that problem. Building on their analysis, we have shown, in Section 5, that the the consequence for estimation of the restriction that B is homothetic in prices is small enough to be ignored.

$\infty \qquad \infty \qquad \infty \qquad \infty$

Table 1

How Big are $\widetilde{\nabla_{\ln x} w}\ \widetilde{w}'$ and $\widetilde{\Phi}$?

Social Welfare Function	Marginal Utility	Preferences	$E\left[abs\left(\frac{dp'\widetilde{\nabla_{\ln x} w}\ \widetilde{w}' dp}{dp'\widetilde{\mathbf{S}}dp}\right)\right]$	$E\left[abs\left(\frac{dp'\widetilde{\Phi}dp}{dp'\widetilde{\mathbf{S}}dp}\right)\right]$
$g(u_i) = u_i$	$h(t) = \exp(t)$	QAI	0.129	0.062
(Utilitarian)	(money-metric)	AI	0.132	0.045
	$h(t) = t$	QAI	0.138	0.022
	(log money-metric)	AI	0.176	0.000
$g(u_i) = \ln u_i$	$h(t) = \exp(t)$	QAI	0.241	0.084
(Inequality-Averse)	(money-metric)	AI	0.127	0.052
	$h(t) = t$	QAI	0.133	0.023
	(log money-metric)	AI	0.206	0.022
$g(u_i) = (u_i)^{-1}$	$h(t) = \exp(t)$	QAI	0.125	0.044
(Strongly Inequality-Averse)	(money-metric)	AI	0.472	0.030
	$h(t) = t$	QAI	0.173	0.070
	(log money-metric)	AI	0.209	0.054

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